Chapter 1

Graphs and Laplacian Matrices

We introduce basic elements of directed graphs, including nodes, edges, subgraphs, neighbors, and degrees. Then graph connectivity concepts key for multi-agent cooperative control problems are introduced; these concepts include strongly connectedness, strong components, spanning trees, and spanning multiple trees. We then introduce relevant matrices of directed graphs, including adjacency matrices, degree matrices, and Laplacian matrices. In particular, we define three types of Laplacian matrices and analyze their algebraic properties (eigenstructures and ranks). Key relations between these algebraic properties of graph matrices and graph connectivity conditions are established.

1.1 Directed graphs

A *directed graph* (or simply *digraph*) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty finite set $\mathcal{V}$ of elements called *nodes*, and a finite set $\mathcal{E}$ of ordered pairs of nodes called *edges*. Thus $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ (the Cartesian product of $\mathcal{V}$ and itself). The set $\mathcal{V}$ is called the *node set* and $\mathcal{E}$ the *edge set* of digraph $\mathcal{G}$.

Three examples of digraphs are displayed in Fig. 1.1:

$$\begin{aligned}
\mathcal{G}_1 &= (\{v_1, v_2, v_3, v_4\}, \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_2), (v_3, v_4), (v_4, v_1), (v_4, v_2)\}) \\
\mathcal{G}_2 &= (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_1, v_3), (v_3, v_2)\}) \\
\mathcal{G}_3 &= (\{v_1, v_2, v_3\}, \{(v_1, v_1), (v_1, v_2), (v_1, v_3), (v_3, v_2)\}).
\end{aligned}$$

For an edge (u, v) the first node u is its *tail* and the second node v is its *head*. The edge (u, v) is said to *leave* u and *enter* v. The head and tail of an edge are its *end-nodes*. A *loop* is an edge whose end-nodes are the same node. An edge is *multiple* if there is another edge with the same end-nodes. A digraph is *simple* if it has no loops or multiple edges.[1]

[1] In this book, unless otherwise specified, only simple digraphs are considered.

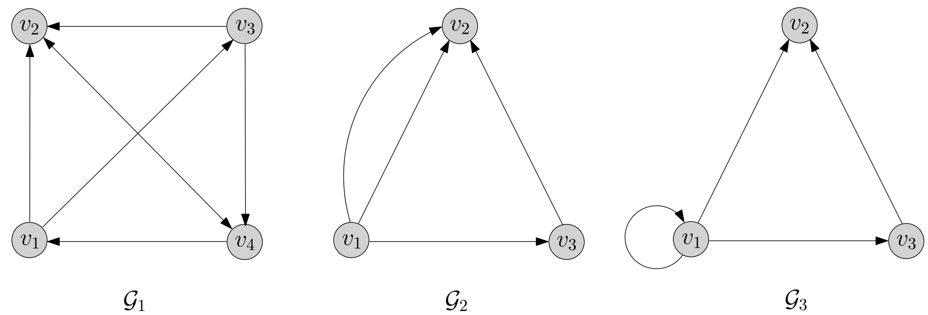


Figure 1.1: Directed graphs (digraphs)

For example, consider the digraphs in Fig. 1.1. Here, digraph $\mathcal{G}_1$ is simple; digraph $\mathcal{G}_2$ has multiple edges, namely (v_1, v_2); and digraph $\mathcal{G}_3$ has a loop, namely (v_1, v_1).

In the special case where for every edge $(u, v) \in \mathcal{E}$, the edge (v, u) of the opposite direction is also an edge, i.e. $(v, u) \in \mathcal{E}$, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an *undirected* graph.

Two examples of undirected graphs are given in Fig. 1.2:

$$\mathcal{G}_1 = (\{v_1, v_2, v_3, v_4\}, \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2), (v_3, v_4), (v_4, v_3), (v_4, v_1), (v_1, v_4)\})$$
$$\mathcal{G}_2 = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_3, v_1)\}).$$

For undirected graphs, their edges are commonly drawn without arrows as in Fig. 1.2.

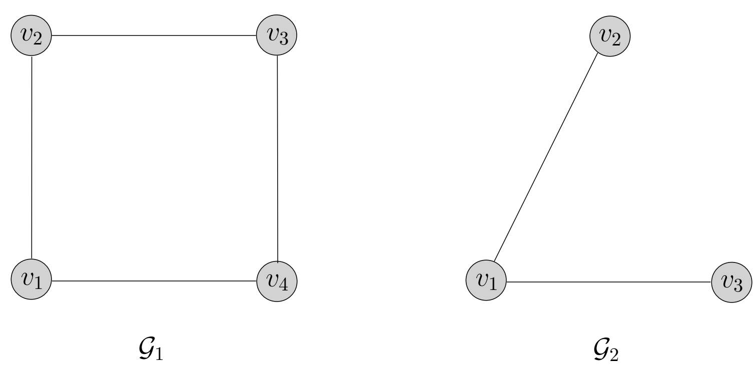


Figure 1.2: Undirected graphs

Subdigraphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph. We say that $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a *subdigraph* of $\mathcal{G}$ if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. If moreover $\mathcal{V}' = \mathcal{V}$, then $\mathcal{G}'$ is a *spanning subdigraph* of $\mathcal{G}$. For a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nonempty subset $\mathcal{V}' \subseteq \mathcal{V}$, the *induced subdigraph* by $\mathcal{V}'$ is $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, with $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}')$.

For example, consider the digraphs displayed in Fig. 1.3. Here $\mathcal{G}_{11}$, $\mathcal{G}_{12}$, and $\mathcal{G}_{13}$ are subdigraphs of $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E})$ in Fig. 1.1. Only $\mathcal{G}_{12}$ is a spanning subdigraph, while only $\mathcal{G}_{13}$ is the induced subdigraph by $\mathcal{V}' = \{v_1, v_2, v_4\} \subseteq \mathcal{V}$. Note that $\mathcal{G}_{11}$ is not the induced subdigraph by $\mathcal{V}' = \{v_1, v_2, v_4\}$ because edge (v_4, v_2) is absent and $\mathcal{E}' \subsetneqq \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}')$.

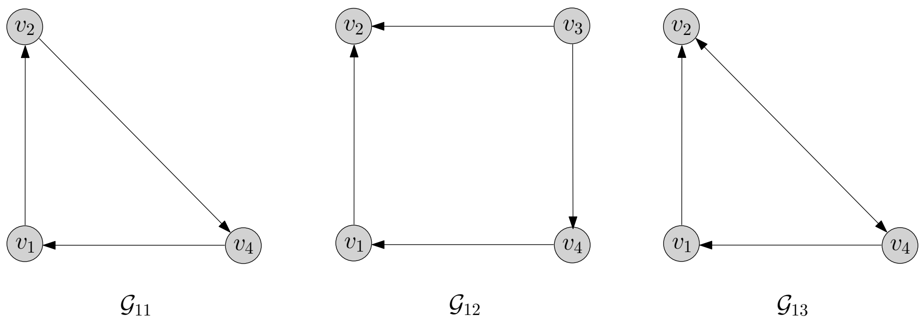


Figure 1.3: Subdigraphs

Neighbors and degrees

The local structure of a digraph is described by the neighbors and the degrees of its nodes. For a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a node $v \in \mathcal{V}$, the *(in-)neighbor set* of v is $\mathcal{N}_v := \{u \in \mathcal{V} \mid (u, v) \in \mathcal{E}\}$, while the *out-neighbor set* of v is $\mathcal{N}_v^o := \{u \in \mathcal{V} \mid (v, u) \in \mathcal{E}\}$. Thus $\mathcal{N}_v$ is a set of nodes that are connected to v with an edge (v being the head), whereas $\mathcal{N}_v^o$ is a set of nodes to which v is connected with an edge (v being the tail). The nodes in $\mathcal{N}_v$ and $\mathcal{N}_v^o$ are respectively the *(in-)neighbors* and *out-neighbors* of v.

The *(in-)degree*, d_v, of a node v is the cardinality of the neighbor set $\mathcal{N}_v$, written $d_v = |\mathcal{N}_v|$. Similarly, the *out-degree*, d_v^o, of a node v is the cardinality of the out-neighbor set $\mathcal{N}_v^o$, i.e. $d_v^o = |\mathcal{N}_v^o|$.

A node v with $d_v = d_v^o$ is called *balanced*. A digraph $\mathcal{G}$ is *balanced* if every node is balanced. Every undirected graph is balanced.

As an illustration, consider the digraph $\mathcal{G}_1$ displayed in Fig. 1.1. For node v_1, its neighbor set is $\mathcal{N}_{v_1} = \{v_4\}$ and out-neighbor set $\mathcal{N}_{v_1}^o = \{v_2, v_3\}$; hence its degree is $d_{v_1} = 1$ and out-degree $d^o_{v_1} = 2$. As a result, v_1 is not balanced. Next consider the digraph $\mathcal{G}_{11}$ in Fig. 1.3. Observe that every node has degree 1 and out-degree 1, so every node is balanced and digraph $\mathcal{G}_{11}$ is balanced.

1.2 Connectivity of digraphs

A (directed) *path* in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a sequence of nodes

$$v_1 v_2 \cdots v_k \quad (k \geq 1)$$

such that $(v_i, v_{i+1}) \in \mathcal{E}$ for every $i = 1, 2, \ldots, k-1$. The path is said to be *from* v_1 *to* v_k. If $v_1 = v_k$, the path is called a *cycle*. The *length* of a path is the number of the consisting edges. Hence the path above has length $k-1$. It is allowed that $k = 1$, in which case the path is of length 0. Also note that a loop (v_i, v_i) is a cycle of length 1.

Let $u, v \in \mathcal{V}$ be two nodes of $\mathcal{G}$. We say that v is *reachable* from u if there is a path from u to v; written $u \to v$. If v is *not* reachable from u, we write $u \not\to v$. Every node v is reachable from itself, i.e. $v \to v$, by the (trivial) path v of length 0. For any node v, the set of nodes reachable from v is

$$\mathcal{V}(v^{\to}) = \{v' \in \mathcal{V} \mid v \to v'\}$$

while the set of nodes from which v is reachable is

$$\mathcal{V}({}^{\to}v) = \{v' \in \mathcal{V} \mid v' \to v\}.$$

We call $\mathcal{V}(v^{\to})$ the *reachable set* of v, and $\mathcal{V}({}^{\to}v)$ the *backward reachable set* of v. Both $\mathcal{V}(v^{\to})$ and $\mathcal{V}({}^{\to}v)$ are nonempty, because v belongs to both.

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *strongly connected* if

$$(\forall u, v \in \mathcal{V}) u \to v$$

namely every node is reachable from every other node. In this case, $\mathcal{V}(v^{\to}) = \mathcal{V}({}^{\to}v) = \mathcal{V}$ for every node $v \in \mathcal{V}$.

For example, consider digraph $\mathcal{G}_1$ in Fig. 1.4. Although for $i = 1, 2, 3$ there holds $\mathcal{V}(v_i^{\to}) = \mathcal{V}({}^{\to}v_i) = \mathcal{V}$, for $i = 4, 5$ only $\mathcal{V}(v_i^{\to}) = \{v_4, v_5\} \subsetneqq \mathcal{V}$. The latter means that nodes v_4, v_5 cannot reach v_1, v_2, v_3. Hence $\mathcal{G}_1$ is not strongly connected. By contrast, $\mathcal{G}_2$ is strongly connected: $\mathcal{V}(v_i^{\to}) = \mathcal{V}({}^{\to}v_i) = \mathcal{V}$ for all $i = 1, 2, 3$.

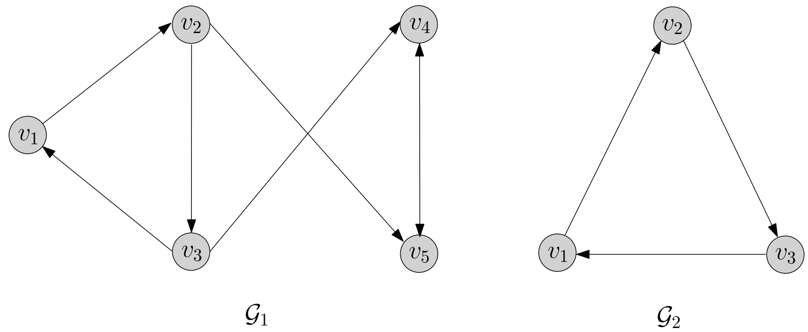


Figure 1.4: Reachability and strongly connected digraphs

A strongly connected digraph $\mathcal{G}$ contains at least one cycle. Given a strongly connected digraph $\mathcal{G}$ containing $m(\geq 1)$ cycles, let $l_1, \ldots, l_m$ be the lengths of these cycles and denote by p their greatest common divisor, i.e.

$$p := \text{g.c.d.}\{l_1, \ldots, l_m\}.$$

If $p > 1$, we say that $\mathcal{G}$ is *periodic* with period p. Otherwise ($p = 1$), we say that $\mathcal{G}$ is *aperiodic*. Note that a strongly connected digraph with a loop is aperiodic (as in this case the loop is a cycle of length 1 and this renders the greatest common divisor $p = 1$).

In a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a node $r \in \mathcal{V}$ is called a *root* if

$$(\forall v \in \mathcal{V}) r \to v$$

that is, every node is reachable from r (equivalently $\mathcal{V}(r^{\to}) = \mathcal{V}$). Note that in a strongly connected digraph $\mathcal{G}$, every node is a root.

Let r be a root of digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A spanning subdigraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ is called a *spanning tree (with root r)* if

- r has no neighbor, i.e. $\mathcal{N}_r = \emptyset$;
- every node $v \in \mathcal{V} \setminus \{r\}$ has exactly one neighbor, i.e. $d_v = 1$.

Definition 1.1 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph. We say that $\mathcal{G}$ contains a spanning tree if there exists a spanning subdigraph of $\mathcal{G}$ that is a spanning tree.*

Note that by definition, $\mathcal{G}$ contains a spanning tree if and only if there exists a root in $\mathcal{G}$.

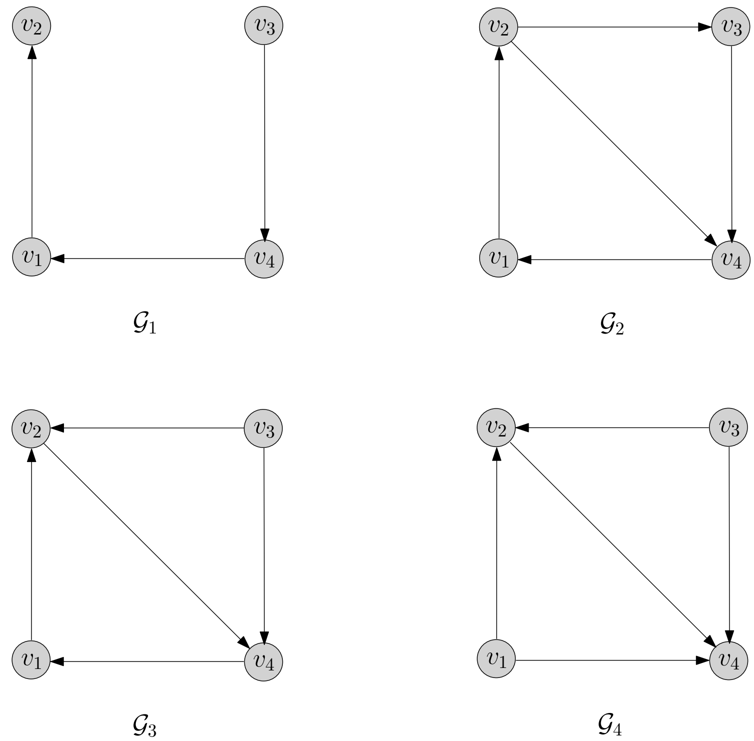


Figure 1.5: Strongly connected digraphs and spanning trees

Consider the digraphs displayed in Fig. 1.5. Digraph $\mathcal{G}_1$ is a spanning tree with root v_3. $\mathcal{G}_2$ is strongly connected, and (so) it contains a spanning tree (say $\mathcal{G}_1$). $\mathcal{G}_3$ is not strongly connected, but contains a spanning tree ($\mathcal{G}_1$). Finally $\mathcal{G}_4$ is neither strongly connected nor contains a spanning tree.

Note that if $\mathcal{G}$ is strongly connected, then $\mathcal{G}$ contains a spanning tree; but the reverse need not hold. Nevertheless whether or not $\mathcal{G}$ contains a spanning tree may be verified by inspecting its strongly connected subdigraphs.

Strong components

Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a subdigraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\emptyset \neq \mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}')$. Namely $\mathcal{G}'$ is an induced subdigraph of $\mathcal{G}$ by $\mathcal{V}'$. We say that $\mathcal{G}'$ is a *strong component* of $\mathcal{G}$ if $\mathcal{G}'$ is strongly connected and for every other induced subdigraph $\mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'')$ with $\mathcal{V}' \subseteq \mathcal{V}''$ and $\mathcal{E}' \subseteq \mathcal{E}''$, $\mathcal{G}''$ is not strongly connected. In other words, $\mathcal{G}'$ is a *maximal* strongly connected induced subdigraph of $\mathcal{G}$ (which need not be unique). Let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two strong components of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then they are either identical (i.e. $\mathcal{V}_1 = \mathcal{V}_2, \mathcal{E}_1 = \mathcal{E}_2$) or disjoint (i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset, \mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$).

A strong component $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is said to be *closed* if

$$(\forall u \in \mathcal{V}')(\forall v \in \mathcal{V} \setminus \mathcal{V}')v \not\to u$$

namely no edge enters any node in $\mathcal{V}'$. In this case, $\mathcal{V}' = \mathcal{V}(^{\to}u) \subseteq \mathcal{V}(u^{\to})$ for every node $u \in \mathcal{V}'$.

Fig. 1.6 provides examples of induced subdigraphs, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, of the first digraph $\mathcal{G}$, where $\mathcal{G}_1$ is not a strong component, $\mathcal{G}_2$ is a closed strong component, and $\mathcal{G}_3$ is a strong component but not closed.

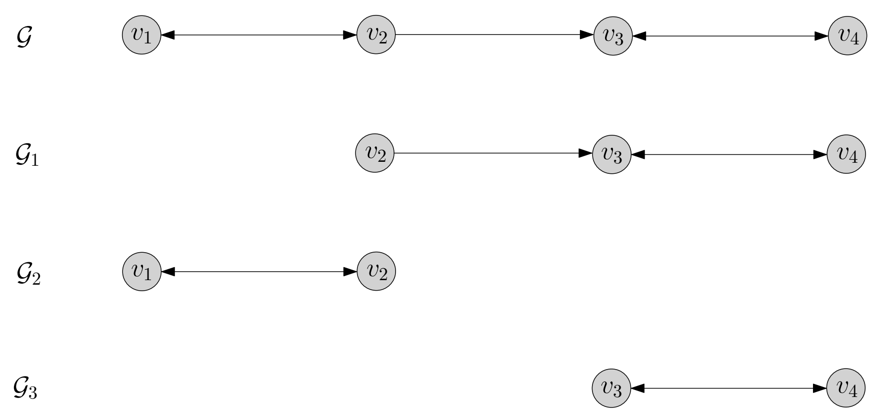


Figure 1.6: Strong components and closed strong components

Theorem 1.1 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph. The following statements are equivalent:*

(i) *$\mathcal{G}$ contains a spanning tree;*

(ii) *$\mathcal{G}$ contains a unique closed strong component.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains a spanning tree. Let $\mathcal{V}_r$ be the subset of all roots, i.e.

$$\mathcal{V}_r := \{r \in \mathcal{V} \mid \mathcal{V}(r^{\to}) = \mathcal{V}\}.$$

Thus $\mathcal{V}_r \neq \emptyset$. Let $\mathcal{G}_r$ be the induced subdigraph by $\mathcal{V}_r$. It will be shown that $\mathcal{G}_r$ is the unique closed strong component of $\mathcal{G}$.

If $\mathcal{V}_r = \mathcal{V}$, namely every node is a root, then $\mathcal{G}_r = \mathcal{G}$ is strongly connected; thus maximality, closedness, and uniqueness follow trivially.

If $\mathcal{V}_r \subsetneqq \mathcal{V}$ (i.e. $\mathcal{V}_r$ is a strict subset of $\mathcal{V}$), first note that $\mathcal{G}_r$ is closed. To see this, suppose on the contrary that there exist $r \in \mathcal{V}_r$ and $v \in \mathcal{V} \setminus \mathcal{V}_r$ such that $v \to r$. Since r is a root, v is also a root, but this contradicts $v \notin \mathcal{V}_r$. Next, note that $\mathcal{G}_r$ is strongly connected. This follows from the fact that every node in $\mathcal{V}_r$ is a root and $\mathcal{G}_r$ is closed. Moreover, no node in $\mathcal{V} \setminus \mathcal{V}_r$ (i.e. non-root) can be added to $\mathcal{V}_r$ while preserving strongly connectedness, so $\mathcal{G}_r$ is a closed strong component of $\mathcal{G}$. Finally, we prove that $\mathcal{G}_r$ is unique. Let $\mathcal{G}'_r = (\mathcal{V}'_r, \mathcal{E}'_r)$ be another closed strong component of $\mathcal{G}$. Then either $\mathcal{V}'_r \cap \mathcal{V}_r = \emptyset$ or $\mathcal{V}'_r = \mathcal{V}_r$. Since all nodes in $\mathcal{V}_r$ are roots, they can reach all nodes in $\mathcal{V}'_r$, but this contradicts closedness of $\mathcal{G}'_r$. Hence, it is only possible that $\mathcal{V}'_r = \mathcal{V}_r$, and $\mathcal{G}'_r = \mathcal{G}_r$ after all. This establishes that $\mathcal{G}_r$ is the unique closed strong component of $\mathcal{G}$.

(ii) $\Rightarrow$ (i). Suppose that $\mathcal{G}$ contains a unique closed strong component $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r)$. We will prove that $\mathcal{G}$ contains a spanning tree by showing that every node in $\mathcal{V}_r$ is a root.

Let $r \in \mathcal{V}_r$ and suppose on the contrary that r is not a root. Then $\mathcal{V}(r^{\rightarrow}) \subsetneqq \mathcal{V}$. Let $\mathcal{U} := \mathcal{V} \setminus \mathcal{V}(r^{\rightarrow})$; thus $\mathcal{U} \neq \emptyset$. Note that no node in $\mathcal{V}(r^{\rightarrow})$ can reach any node in $\mathcal{U}$, because otherwise r could also reach some node in $\mathcal{U}$. Hence the induced subdigraph $\mathcal{G}_u$ by $\mathcal{U}$ is closed. In the following, it will be shown that $\mathcal{G}_u$ contains at least one closed strong component.

Select an arbitrary node $u_1 \in \mathcal{U}$, and check if $\mathcal{V}(^{\rightarrow}u_1) \subseteq \mathcal{V}(u_1^{\rightarrow})$. If so, it follows that the induced subdigraph $\mathcal{G}_1$ by $\mathcal{V}(^{\rightarrow}u_1)$ is a closed strong component of $\mathcal{G}_u$. If the condition fails, then select another arbitrary node $u_2 \in \mathcal{V} \setminus \mathcal{V}(^{\rightarrow}u_1)$, and check if $\mathcal{V}(^{\rightarrow}u_2) \subseteq \mathcal{V}(u_2^{\rightarrow})$. Note that here $\mathcal{V}(^{\rightarrow}u_2) \subseteq \mathcal{V} \setminus \mathcal{V}(^{\rightarrow}u_1)$ necessarily holds, for otherwise u_1 could be reached from u_2. If the condition holds, then the induced subdigraph $\mathcal{G}_2$ by $\mathcal{V}(^{\rightarrow}u_2)$ is a closed strong component of $\mathcal{G}_u$. If not, repeat the above procedure. Since the node set $\mathcal{U}$ is finite, in the worst case after (say) k repetitions and check failures, the subset $\mathcal{V}(^{\rightarrow}u_{k+1}) \subseteq \mathcal{V} \setminus \mathcal{V}(^{\rightarrow}u_1) \setminus \cdots \setminus \mathcal{V}(^{\rightarrow}u_k)$ contains a singleton node u_{k+1}. Since $\mathcal{V}(^{\rightarrow}u_{k+1}) \subseteq \mathcal{V}(u_{k+1}^{\rightarrow})$ holds trivially, the induced subdigraph $\mathcal{G}_{k+1}$ by $\mathcal{V}(^{\rightarrow}u_{k+1})$ is a closed strong component of $\mathcal{G}_u$.

We have thus proved that $\mathcal{G}_u$ contains a closed strong component, say $\mathcal{G}'_u$. Since $\mathcal{G}_u$ is closed in $\mathcal{G}$, $\mathcal{G}'_u$ is also a closed strong component of $\mathcal{G}$. But $\mathcal{G}'_u$ is different from $\mathcal{G}_r$, which is a contradiction to the assumed uniqueness of the strong component $\mathcal{G}_r$. Therefore, every node $r \in \mathcal{V}_r$ is a root and $\mathcal{G}$ contains at least one spanning tree. □

To illustrate Theorem 1.1, consider the digraphs in Fig. 1.4. $\mathcal{G}_1$ contains two strong components, but only the one induced by $\{v_1, v_2, v_3\}$ is closed. Hence $\mathcal{G}_1$ has a unique closed strong component, and therefore $\mathcal{G}_1$ contains a spanning tree with root (say) v_1. $\mathcal{G}_2$ contains only one strong component, namely itself, which is (trivially) closed. So again $\mathcal{G}_2$ contains a spanning tree with root (say) v_3. On the other hand, consider digraph $\mathcal{G}_4$ in Fig. 1.5. We have identified that $\mathcal{G}_4$ does not contain a spanning tree. Indeed, this digraph contains 4 strong components, two of which are closed: one induced by $\{v_1\}$ and the other by $\{v_3\}$. Namely $\mathcal{G}_4$ fails to have a unique closed strong component.

Spanning multiple trees

Let us now generalize the concept of spanning trees to allow multiple roots.

Consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\mathcal{R} \subseteq \mathcal{V}$ be a subset of nodes, and $k := |\mathcal{R}|$. Consider $k \geq 2$, i.e. $\mathcal{R}$ contains at least two nodes. Let $v \in \mathcal{V} \setminus \mathcal{R}$. We say that v is *k-reachable* from $\mathcal{R}$ if there is a path from a node in $\mathcal{R}$ to v after removing arbitrary $k-1$ nodes except for v itself; written $\mathcal{R} \rightarrow_k v$. More formally, $\mathcal{R} \rightarrow_k v$ if

$$(\forall \mathcal{U} \subseteq \mathcal{V} \setminus \{v\}) |\mathcal{U}| = k-1 \Rightarrow (\exists r \in \mathcal{R} \cap (\mathcal{V} \setminus \mathcal{U})) r \rightarrow v \text{ in } \mathcal{G}' \text{ induced by } \mathcal{V} \setminus \mathcal{U}.$$

If v is *not* k-reachable from $\mathcal{R}$, we write $\mathcal{R} \not\rightarrow_k v$.

The subset $\mathcal{R}$ of $k (\geq 2)$ nodes is called a *k-root subset* if

$$(\forall v \in \mathcal{V} \setminus \mathcal{R}) \mathcal{R} \rightarrow_k v$$

that is, every node (not in $\mathcal{R}$) is k-reachable from $\mathcal{R}$. Note that in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, if $\mathcal{R}$ is a k-root subset, then for every $r \in \mathcal{R}$, $\mathcal{R} \setminus \{r\}$ is a $(k-1)$-root subset in the induced subgraph by $\mathcal{V} \setminus \{r\}$. In the special case $k = 2$, i.e. $\mathcal{R} = \{r_1, r_2\}$, r_1 (resp. r_2) is a root of the induced subgraph by $\mathcal{V} \setminus \{r_2\}$ (resp. by $\mathcal{V} \setminus \{r_1\}$).

Consider the digraphs in Fig. 1.7. In $\mathcal{G}_1$, v_1 is 2-reachable from $\{v_2, v_3\}$, and $\{v_2, v_3\}$ is a 2-root set. By contrast, in $\mathcal{G}_2$, v_1 is not 2-reachable from $\{v_2, v_3\}$, because after removing v_2, v_1 is no longer reachable from v_3. Similarly, in $\mathcal{G}_3$, v_1 is 3-reachable from $\{v_2, v_3, v_4\}$, and $\{v_2, v_3, v_4\}$ is a 3-root set. But in $\mathcal{G}_4$, v_1 is not 3-reachable, because after removing v_2 and v_3, v_1 is not reachable from v_4. Finally, removing v_2 in $\mathcal{G}_1$, v_3 is a root of the induced subgraph by $\{v_1, v_3\}$; also removing v_4 in $\mathcal{G}_3$, $\{v_2, v_3\}$ is a 2-root subset of the induced subgraph by $\{v_1, v_2, v_3\}$.

Let $\mathcal{R}$ be a k-root subset of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A spanning subdigraph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ is called a *spanning k-tree (with k-root subset $\mathcal{R}$)* if

- every root $r \in \mathcal{R}$ has no neighbor, i.e. $\mathcal{N}_r = \emptyset$;
- every node $v \in \mathcal{V} \setminus \mathcal{R}$ has exactly k neighbors, i.e. $d_v = k$.

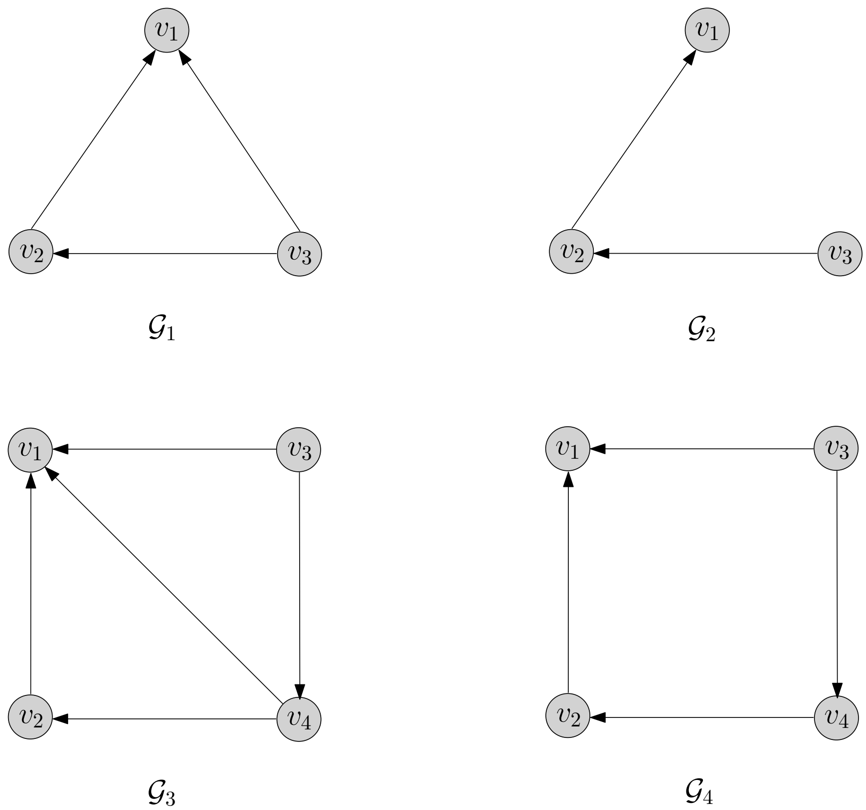


Figure 1.7: k-reachability

Definition 1.2 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph and $k \geq 2$. We say that $\mathcal{G}$ contains a spanning k-tree if there exists a spanning subdigraph of $\mathcal{G}$ that is a spanning k-tree.*

By this definition, $\mathcal{G}$ contains a spanning k-tree if and only if there exists a k-root subset in $\mathcal{G}$.

As an illustration, $\mathcal{G}_1$ in Fig. 1.7 contains a spanning 2-tree $\mathcal{G}_1'$, which is displayed in Fig. 1.8. For another example, $\mathcal{G}_3$ in Fig. 1.7 contains a spanning 3-tree $\mathcal{G}_2'$ in Fig. 1.8.

A counterpart of Theorem 1.1 is the following, which establishes the relation between $\mathcal{G}$ containing a spanning k-tree and the number of closed strong components.

Theorem 1.2 *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph and $k \geq 2$. If $\mathcal{G}$ contains a spanning k-tree, then $\mathcal{G}$ contains $l \in [1, k]$ closed strong components.*

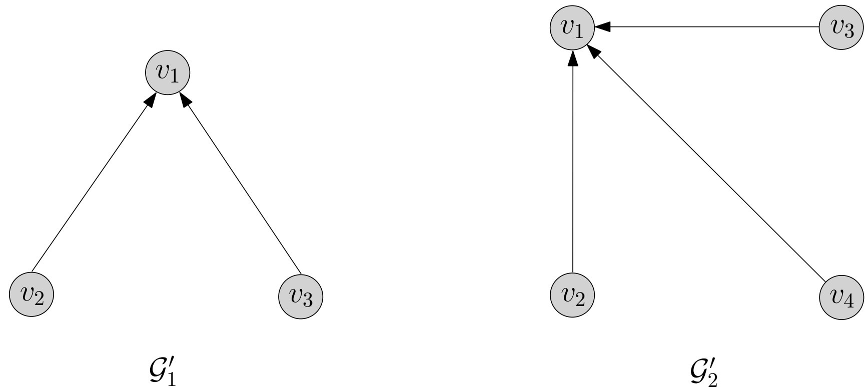


Figure 1.8: Spanning k-tree

Proof. Suppose on the contrary that $\mathcal{G}$ contains $k+1$ closed strong components: $\mathcal{G}_1, \ldots, \mathcal{G}_k, \mathcal{G}_{k+1}$. It will be shown that there cannot exist a k-root subset, and consequently $\mathcal{G}$ does not contain a spanning k-tree.

Consider an arbitrary subset $\mathcal{V}'$ of k nodes in $\mathcal{G}$. Since there are $k+1$ closed strong components, there exists at least one closed strong component $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ $(i \in [1, k+1])$ such that $\mathcal{V}' \cap \mathcal{V}_i = \emptyset$. Namely $\mathcal{G}_i$ does not contain any node in $\mathcal{V}'$. Now choose a node v_i in $\mathcal{G}_i$, so $v_i \in \mathcal{V}_i$ and $v_i \notin \mathcal{V}'$. Then remove $k-1$ nodes from the other k closed strong components ($\mathcal{G}_i$ excluded). Since $\mathcal{G}_i$ is closed, the chosen node v_i cannot be reached from the subset $\mathcal{V}'$. This by definition means that $\mathcal{V}'$ is not a k-root subset. Since $\mathcal{V}'$ is arbitrary, we conclude that there cannot exist a k-root subset in $\mathcal{G}$. This completes the proof. □

To illustrate Theorem 1.2, first consider $k = 2$. Both $\mathcal{G}_1$ in Fig. 1.7 and $\mathcal{G}'_1$ in Fig. 1.8 contain a spanning 2-tree. While $\mathcal{G}_1$ contains 1 closed strong component (induced by $\{v_3\}$), $\mathcal{G}'_1$ contains 2 closed strong components (induced respectively by $\{v_2\}$ and $\{v_3\}$). Next consider $k = 3$. The digraphs in Fig. 1.9 contain a spanning 3-tree. $\mathcal{G}'_3$ has 1 closed strong component (induced by $\{v_2, v_3, v_4\}$), while $\mathcal{G}'_4$ has 2 closed strong components (induced respectively by $\{v_2, v_4\}$ and $\{v_3\}$). In addition, the spanning 3-tree $\mathcal{G}'_2$ in Fig. 1.8 has 3 closed strong components (induced respectively by $\{v_2\}$, $\{v_3\}$, and $\{v_4\}$).

1.3 Matrices of digraphs

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \ldots, v_n\}$, we may assign to each edge $(v_j, v_i) \in \mathcal{E}$ a *weight* a_{ij}. If a pair (v_j, v_i) is not an edge, i.e. $(v_j, v_i) \notin \mathcal{E}$, then $a_{ij} = 0$. The weight a_{ij} may be a positive

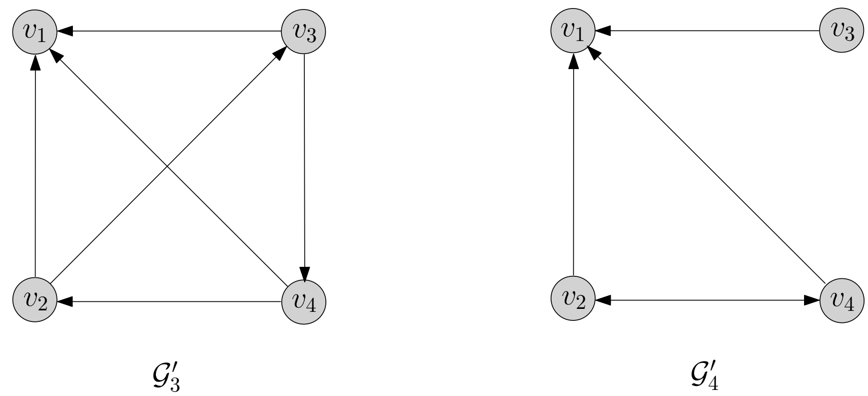


Figure 1.9: Number of closed strong components in digraphs containing a spanning multiple tree

real number, or any real number, or even a complex number. We note that even if $(v_j, v_i) \in \mathcal{E}$, the corresponding weight a_{ij} can still be 0, i.e. an edge can have zero weight. In this case, it is equivalent to treat such a zero-weight edge as nonexisting in the digraph. With weights assigned to edges, the digraph $\mathcal{G}$ is called a *weighted digraph*.

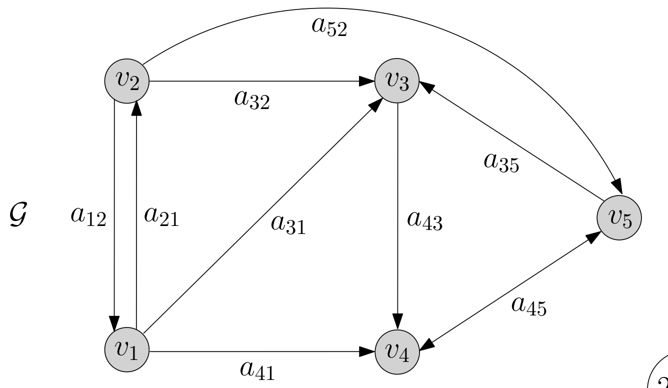
The *adjacency matrix* of a weighted digraph $\mathcal{G}$ is an $n \times n$ matrix $A = (a_{ij})$. Depending on the field where a_{ij} belongs, A may be a nonnegative matrix (entry-wise nonnegative) if $a_{ij} > 0$, an arbitrary real matrix if $a_{ij} \in \mathbb{R}$, or a complex matrix if $a_{ij} \in \mathbb{C}$. In the case that the adjacency matrix A is nonnegative, $a_{ij} > 0$ if and only if $(v_j, v_i) \in \mathcal{E}$.

Conversely for a given $n \times n$ matrix $A = (a_{ij})$, we may construct a weighted digraph $\mathcal{G}(A)$ of n nodes such that an edge (v_j, v_i) exists with weight a_{ij} if and only if $a_{ij} \neq 0$.

Illustration of adjacency matrices is provided in Fig. 1.10. Given a weighted digraph $\mathcal{G}$ of five nodes, its adjacency matrix A is a 5×5 matrix with each entry a_{ij} the weight on edge (v_j, v_i). Conversely for a given 4×4 matrix A', its corresponding digraph $\mathcal{G}(A')$ has four nodes, and an edge (v_j, v_i) with weight a_{ij} exists whenever $a_{ij} \neq 0$. Note that the two loops in $\mathcal{G}(A')$ are due to the nonzero diagonal entries a_{11} and a_{44}.

We write $A \geq 0$ if A is a nonnegative matrix, and $A > 0$ if A is a positive matrix (entry-wise positive). The same notation is used for nonnegative and positive vectors (which are special one-column matrices).

When the adjacency matrix A is a nonnegative matrix (i.e. $A \geq 0$), there are several important properties concerning its *spectrum* (i.e. set of eigenvalues) that we shall introduce in the sequel (the Perron-Frobenius Theorem in Theorem 1.5). To this end, we introduce two types of nonnegative matrices in order: irreducible matrices and primitive matrices.



$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & a_{35} \\ a_{41} & 0 & a_{43} & 0 & a_{45} \\ 0 & a_{52} & 0 & a_{54} & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -0.5 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

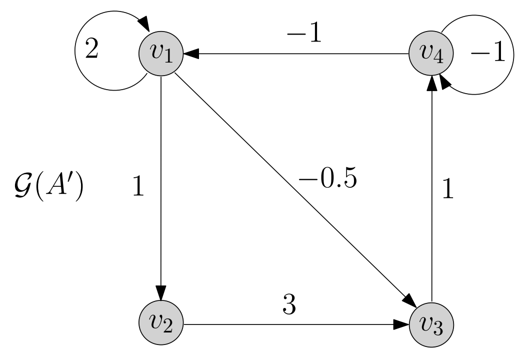


Figure 1.10: Adjacency matrices

Irreducible matrices

A square matrix P is a *permutation matrix* if for each row and each column, there is exactly one entry equal to 1. That is, the columns of a permutation matrix are a reordering of the standard basis vectors. Indeed, if P is a permutation matrix and M an arbitrary matrix, then the operation $M \mapsto PM$ amounts to reordering the rows of M; further $PM \mapsto PMP^\top$ amounts to doing the same reordering of the columns of PM. A permutation matrix P is *orthogonal*: $P^\top P = PP^\top = I$.

Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, i.e. $A \geq 0$. We say that A is *reducible* if either (i) $n = 1$ and $A = 0$, or (ii) there exists a permutation matrix P such that $PAP^\top$ is block upper triangular as follows:

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and D are square matrices. Otherwise A is *irreducible*.

For example, consider two nonnegative matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \end{bmatrix}.$$

A_1 is reducible because there exists the following permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{such that} \quad PA_1P^\top = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 5 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

On the other hand, A_2 is irreducible: no permutation matrix P can render $PA_2P^\top$ in a block upper triangular form.

Irreducibility of matrices is elegantly characterized by connectivity of digraphs.

Theorem 1.3 *Let $\mathcal{G}$ be a weighted digraph with n nodes and $A \geq 0$ the corresponding nonnegative adjacency matrix. Then A is irreducible if and only if $\mathcal{G}$ is strongly connected.*

For the example A_1, A_2 above, they are respectively the nonnegative adjacency matrices of digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Fig. 1.11. A_1 is reducible and digraph $\mathcal{G}_1$ is not strongly connected; whereas A_2 is irreducible and digraph $\mathcal{G}_2$ is strongly connected.

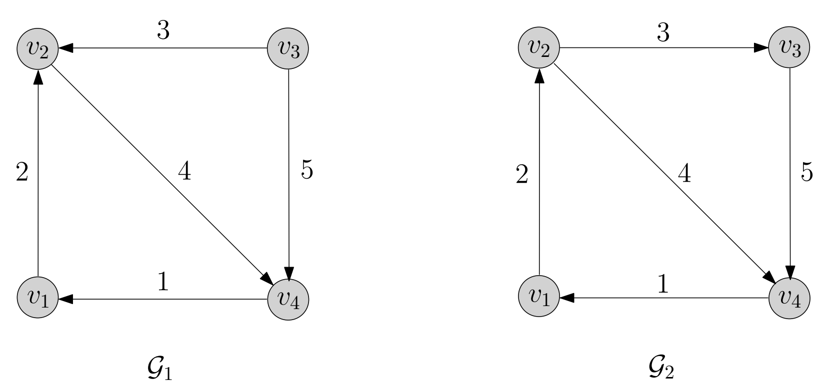


Figure 1.11: Irreducibility of nonnegative matrices characterized by graph connectivity

To prove Theorem 1.3, the following lemma is useful, which establishes a link between positivity of entries in powers of an adjacency matrix and reachability of the corresponding nodes. For an arbitrary positive integer $k \geq 1$, denote by a_{ij}^k the (i,j)-entry of the matrix A^k.

Lemma 1.1 *Let $\mathcal{G}$ be a weighted digraph with n nodes and $A \geq 0$ the corresponding nonnegative adjacency matrix. Then for every $i, j \in \{1, \ldots, n\}$ and every positive integer $k \geq 1$, $a_{ij}^k > 0$ if and only if there exists a path of length k from node v_j to node v_i.*

Proof. The proof is by induction on $k \geq 1$. For the base case where $k = 1$, the assertion holds by the definition of nonnegative adjacency matrix A. Namely, $a_{ij} > 0$ if and only if there is an edge $(v_j, v_i) \in \mathcal{E}$ (i.e. path of length 1 from v_j to v_i).

For the induction step, suppose that the assertion holds for $k-1$. Note from $A^k = A^{k-1}A$ that

$$a_{ij}^k = \sum_{m=1}^{n} a_{im}^{k-1} a_{mj}.$$

Thus $a_{ij}^k > 0$ if and only if there is $m \in \{1, \ldots, n\}$ such that $a_{im}^{k-1} > 0$ and $a_{mj} > 0$. That is, there exist a path of length $k-1$ from node v_m to v_i and a path of length 1 from v_j to v_m. These two paths constitute a path of length k from v_j to v_i. This finishes the induction step, and thereby establishes the assertion for any positive integer $k \geq 1$. □

Proof of Theorem 1.3. (If) Suppose on the contrary that A is reducible. By definition, there is a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} =: \tilde{A}.$$

Then the matrix $I + \tilde{A}$ is also block upper triangular, and so is its $n-1$ powers $(I + \tilde{A})^{n-1}$. Consequently $(I + \tilde{A})^{n-1}$ is not a positive matrix. Note that

$$(I + \tilde{A})^{n-1} = P(I + A)^{n-1}P^\top$$

so neither is $(I + A)^{n-1}$ positive. Since in general

$$(I + A)^{n-1} = I + c_1 A + c_2 A^2 + \cdots + c_{n-1}A^{n-1}$$

and the combinatorial coefficients $c_1, \ldots, c_{n-1}$ are all positive, there exist $i, j \in \{1, \ldots, n\}$ $(i \neq j)$ such that for every $k \in \{1, \ldots, n-1\}$ it holds that $a_{ij}^k = 0$. But this means (by Lemma 1.1) that there is no path of any length $k \in \{1, \ldots, n-1\}$ from node v_j to node v_i. Namely $v_j \not\to v_i$; hence digraph $\mathcal{G}$ is not strongly connected.

(Only if) Suppose on the contrary that $\mathcal{G}$ is not strongly connected. By definition, there exist two nodes v_i, v_j such that $v_j \not\to v_i$. Thus the set of nodes that cannot reach v_i is nonempty, i.e. $\mathcal{V} \setminus \mathcal{V}(^{\to}v_i) \neq \emptyset$. In fact, there does not exist any path from any node in $\mathcal{V} \setminus \mathcal{V}(^{\to}v_i)$ to any node in $\mathcal{V}(^{\to}v_i)$. To see this, suppose that there exist $v_l \in \mathcal{V} \setminus \mathcal{V}(^{\to}v_i)$ and $v_m \in \mathcal{V}(^{\to}v_i)$ such that $v_l \to v_m$. Since $v_m \to v_i$, we have $v_l \to v_i$, but this contradicts $v_l \notin \mathcal{V}(^{\to}v_i)$. By this fact, we reorder the nodes according to the partition of the node set: $\{\mathcal{V} \setminus \mathcal{V}(^{\to}v_i), \mathcal{V}(^{\to}v_i)\}$. The reordering amounts to a permutation of the indices of nodes, and correspondingly there is a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

But this means that A is reducible. □

Primitive matrices

Next we introduce primitive matrices. Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix, i.e. $A \geq 0$. We say that A is *primitive* if

$$(\exists k \geq 1) A^k > 0.$$

A primitive matrix is irreducible, but the converse need not hold. This is evident from the following graphical characterization of primitive matrices, as compared to that of irreducible matrices in Theorem 1.3.

Theorem 1.4 *An $n \times n$ nonnegative matrix A is primitive if and only if $\mathcal{G}(A)$ is strongly connected and aperiodic.*

Consider again the matrix A_2 which is the adjacency matrix of digraph $\mathcal{G}_2$ in Fig. 1.11. We have analyzed that A_2 is irreducible, as $\mathcal{G}_2$ is strongly connected. Moreover $\mathcal{G}_2$ is aperiodic: there are two cycles in $\mathcal{G}_2$ of length 3 and 4, respectively; hence $p = \text{g.c.d.}\{3, 4\} = 1$. By Theorem 1.4, A_2 is primitive. Indeed, it is checked that A_2^{10} is a positive matrix.

Let us consider two more matrices

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}.$$

First, A_3 is not primitive because digraph $\mathcal{G}(A_3)$ in Fig. 1.12 is not aperiodic. Indeed $\mathcal{G}(A_3)$ is a strongly connected digraph of period 4. Hence A_3 is irreducible but not primitive. On the other hand, A_4 is the same as A_3 except for the positive $(1,1)$ entry. This diagonal entry is crucial, however, since digraph $\mathcal{G}(A_4)$ in Fig. 1.12 is aperiodic due to the loop at v_1. Therefore A_4 is primitive (in fact $A_4^6 > 0$).

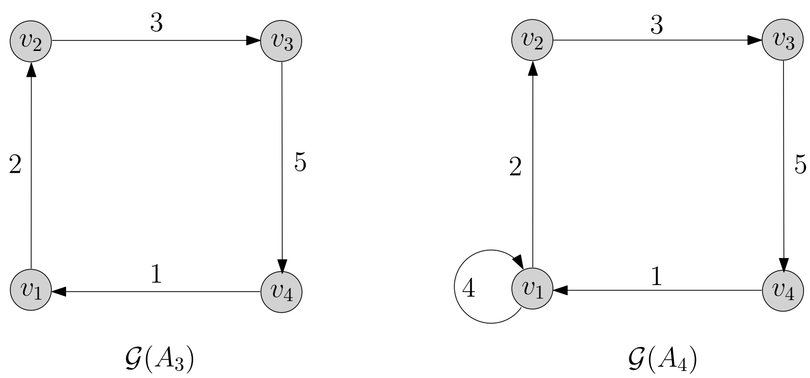


Figure 1.12: Primitivity of nonnegative matrices characterized by graph connectivity

The proof of Theorem 1.4 requires the following lemmas.

Lemma 1.2 *Let $m_1, m_2 \geq 1$ be two positive integers. If* g.c.d.$\{m_1, m_2\} = 1$*, then there is an integer $\bar{k} \geq 0$ such that for any integer $k \geq \bar{k}$,*

$$k = \alpha m_1 + \beta m_2$$

for some nonnegative integers α, β.

Proof. Since

$$\text{g.c.d.}\{m_1, m_2\} = 1,$$

1 is an integer combination of m_1 and m_2, i.e.

$$1 = \alpha_1 m_1 - \beta_1 m_2$$

for some nonnegative integers α_1, β_1. Let $\bar{k} := \beta_1 m_2^2$. Thus $\bar{k} \geq 0$ and for all $k \geq \bar{k}$,

$$k = \beta_1 m_2^2 + i m_2 + j$$

for some integers i, j satisfying $i \geq 0$ and $0 \leq j < m_2$. Substituting $1 = \alpha_1 m_1 - \beta_1 m_2$ into the above equation yields

$$\begin{aligned} k &= \beta_1 m_2^2 + i m_2 + j(\alpha_1 m_1 - \beta_1 m_2) \\ &= (j\alpha_1) m_1 + (\beta_1(m_2 - j) + i)\, m_2. \end{aligned}$$

Let

$$\alpha := j\alpha_1 \text{ and } \beta := \beta_1(m_2 - j) + i.$$

Then α, β are nonnegative integers due to $j < m_2$. Therefore, the conclusion follows. □

The next result shows the relationship between the period of a strongly connected digraph and the period of each node in the digraph. For an arbitrary node v in a strongly connected digraph $\mathcal{G}$, let $l_{v,1}, \ldots, l_{v,m}$ be the lengths of all $m(\geq 1)$ cycles from v to v. Denote by p_v their greatest common divisor, i.e.

$$p_v := \text{g.c.d.}\{l_{v,1}, \ldots, l_{v,m}\}$$

and we say that p_v is the period of node v.

Lemma 1.3 *Consider a strongly connected digraph $\mathcal{G}$. Let p be the period of a digraph $\mathcal{G}$ and p_i be the period of node v_i, $i \in \{1, \ldots, n\}$. Then $p = p_1 = \cdots = p_n$.*

Proof. Let $i \in \{1, \ldots, n\}$. We will establish $p = p_i$ by showing that p divides p_i and p_i divides p.

First let $\mathcal{L} := \{l_1, \ldots, l_k\}$ be the set of all the lengths of all $k(\geq 1)$ cycles in digraph $\mathcal{G}$. Then by definition, p is the greatest common divisor of the elements in $\mathcal{L}$. Note that for every path from v_i to v_i, it is either a (simple) cycle or consists of a number of cycles. So the length l_{v_i} of any path from v_i to v_i is an integer combination of l_j, $j \in \{1, \ldots, k\}$, with nonnegative integer coefficients. This means that every $l_j \in \mathcal{L}$ divides l_{v_i}. Therefore p divides l_{v_i}, which further implies p divides p_i.

On the other hand, consider an arbitrary cycle in digraph $\mathcal{G}$, and let its length be $l_j \in \mathcal{L}$. If the cycle goes through v_i, then p_i divides l_j. If not, then the cycle necessarily goes through some other node, say v_m. Since $\mathcal{G}$ is strongly connected, there must exist a cycle going through v_i and v_m. Denote by $l_{i,m}$ the length of this cycle. Thus p_i divides $l_{i,m}$. Note that these two cycles constitute a path of length $l_{i,m} + l_j$ from v_i to v_i. So p_i divides $l_{i,m} + l_j$ and therefore p_i divides l_j. Hence, p_i divides any l_j in $\mathcal{L}$. This means that p_i divides p.

Based on the above established two facts that p_i divides p and p divides p_i, we conclude that $p = p_i$ for every $i \in \{1, \ldots, n\}$. □

Lemma 1.4 *Let A be an $n \times n$ nonnegative matrix. If $\mathcal{G}(A)$ is strongly connected and p-periodic, then $a_{ii}^k = 0$ for any $i \in \{1, \dots, n\}$ and for any k that is not a multiple of p.*

Proof. Let p_i, $i \in \{1, \dots, n\}$, be the period of the node v_i in $\mathcal{G}(A)$. Thus by Lemma 1.3

$$p = p_1 = \cdots = p_n$$

since $\mathcal{G}(A)$ is strongly connected. Hence the length of any path from v_i to v_i is a multiple of p. Namely there is no path from v_i to v_i with length k that is not a multiple of p. So it follows from Lemma 1.1 that $a_{ii}^k = 0$ for every $i \in \{1, \dots, n\}$ and any k that is not a multiple of p. □

With the three lemmas above, we present the proof of Theorem 1.4.

Proof of Theorem 1.4. (If) Since $\mathcal{G}(A)$ is strongly connected and aperiodic, by Lemma 1.3 the period of $\mathcal{G}(A)$ and the period of each node v_i are equal to 1. For any node v_i, let $l_{v_i}^1, l_{v_i}^2$ ($l_{v_i}^1 \neq l_{v_i}^2$) be the lengths of two paths from v_i to v_i. By Lemma 1.2 there is sufficiently large $\bar{k}_i$ such that for any $k \geq \bar{k}_i$, k may be expressed by a nonnegative integer combination of $l_{v_i}^1$ and $l_{v_i}^2$, which means that there is a path of length k from v_i to v_i. Let v_j be another node. Since $\mathcal{G}(A)$ is strongly connected, there is a path from v_i to v_j; let its length be l_{ij}. Thus for any $k \geq q_{ij} := \bar{k}_i + l_{ij}$ there is a path of length k from v_i to v_j. It follows from Lemma 1.1 that $a_{ij}^k > 0$ for all $k \geq q_{ij}$. Let

$$q := \max\{q_{ij} \mid i, j = 1, \dots, n\}.$$

Then we have $a_{ij}^k > 0$ for all $i, j = 1, \dots, n$ and $k \geq q$. Therefore by definition, A is a primitive matrix.

(Only if) Suppose on the contrary that $\mathcal{G}(A)$ is not strongly connected, or that it is strongly connected but not aperiodic. For the first case that $\mathcal{G}(A)$ is not strongly connected, there is a pair of nodes v_i and v_j such that v_j is not reachable from v_i. So by Lemma 1.1, $a_{ij}^k = 0$ for all $k > 0$. Hence there is no positive integer k such that A^k is positive and consequently A is not primitive.

For the second case, $\mathcal{G}(A)$ is strongly connected but not aperiodic, that is, it is p-periodic where $p > 1$. It follows from Lemma 1.4 that $a_{ii}^{k'} = 0$ for any positive integer k' that is not a multiple of p. Hence there is no positive integer k such that A^k is positive, as otherwise if there were a positive integer k^* such that A^{k^*} is positive, then A^k is positive for any $k \geq k^*$, which contradicts $a_{ii}^{k'} = 0$ for any positive integer k' that is not a multiple of p. Therefore, A is not primitive. □

Perron-Frobenius Theorem

We are now ready to introduce the Perron-Frobenius Theorem. Denote by $\sigma(A)$ the *spectrum* of matrix A, i.e. the set of all eigenvalues of A, and $\rho(A)$ the *spectral radius* of A, i.e. the maximum magnitude of the eigenvalues of A.

Theorem 1.5 (Perron-Frobenius Theorem) *Consider a nonnegative matrix A. If A is irreducible, then*

- $\rho(A) > 0$;
- $\rho(A)$ *is a simple eigenvalue of* A;
- $\rho(A)$ *has a positive eigenvector and a positive left-eigenvector.*[a]

Moreover, if A is primitive, then all eigenvalues except for $\rho(A)$ have absolute values smaller than $\rho(A)$:

- $(\forall \lambda \in \sigma(A))\lambda \neq \rho(A) \Rightarrow |\lambda| < \rho(A)$.

[a] Left-eigenvector w corresponding to an eigenvalue λ of A satisfies $w^\top A = w^\top \lambda$.

Of particular interest is specialization of the Perron-Frobenius Theorem to a special class of nonnegative matrices: *stochastic matrices*. A nonnegative matrix A is called *row-stochastic* (resp. *column-stochastic*) if every row (resp. every column) of A sums up to one; if A is both row-stochastic and column-stochastic, it is called *doubly-stochastic*.

Lemma 1.5 *If A is a row-stochastic (column-stochastic, doubly-stochastic) matrix, then* $\rho(A) = 1$.

Proof. We prove the statement for row-stochastic matrices; the proofs for column-stochastic and doubly-stochastic matrices are similar.

Since A is row-stochastic, we have $A\mathbf{1} = \mathbf{1}$. This means that 1 is an eigenvalue of A. Hence $\rho(A) \geq 1$. On the other hand,

$$
\begin{aligned}
\rho(A) &= \max\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\} \\
&= \max\{\|\lambda x\|_\infty \mid \lambda \text{ is an eigenvalue of } A, x \text{ is a corresponding eigenvector, } \|x\|_\infty = 1\} \\
&= \max\{\|Ax\|_\infty \mid x \text{ is an eigenvector of } A, \|x\|_\infty = 1\} \\
&\leq \max\{\|Ax\|_\infty \mid \|x\|_\infty = 1\} \\
&= \|A\|_\infty \\
&= \max_i \sum_j |a_{ij}| = 1.
\end{aligned}
$$

The last equality follows from the fact that every row of A sums to one. Therefore $\rho(A) = 1$. □

Theorem 1.6 (Perron-Frobenius Theorem for Stochastic Matrices) *Consider a row-stochastic (column-stochastic, doubly-stochastic) matrix A. If A is irreducible, then $\rho(A) = 1$ is a simple eigenvalue of A, with a positive eigenvector and a positive left-eigenvector. Specifically:*

- *if A is row-stochastic, then eigenvalue 1 has a positive eigenvector $\mathbf{1}$ ($A\mathbf{1} = \mathbf{1}$) and a positive left eigenvector π_l ($\pi_l^\top A = \pi_l^\top$);*
- *if A is column-stochastic, then eigenvalue 1 has a positive eigenvector π_r ($A\pi_r = \pi_r$) and a positive left eigenvector $\mathbf{1}$ ($\mathbf{1}^\top A = \mathbf{1}^\top$);*
- *if A is doubly-stochastic, then eigenvalue 1 has a positive eigenvector $\mathbf{1}$ ($A\mathbf{1} = \mathbf{1}$) and a positive left eigenvector $\mathbf{1}$ ($\mathbf{1}^\top A = \mathbf{1}^\top$).*

Moreover, if A is primitive, then all eigenvalues except for 1 have absolute values smaller than 1:

- $(\forall \lambda \in \sigma(A))\lambda \neq 1 \Rightarrow |\lambda| < 1$.

Laplacian matrices

For a weighted digraph $\mathcal{G}$, the *weighted (in-)degree* d_i of a node i is the sum of the weights of all edges entering i, i.e. $d_i = \sum_{j=1}^{n} a_{ij}$. Similarly, the *weighted out-degree* d_i^o of a node i is the sum of the weights of all edges leaving i, i.e. $d_i^o = \sum_{j=1}^{n} a_{ji}$. A node i with $d_i = d_i^o$ is called *weight-balanced*. A digraph $\mathcal{G}$ is *weight-balanced* if every node is weight-balanced.

The *degree matrix* of a weighted digraph $\mathcal{G}$ is $D := \mathrm{diag}(d_1, \ldots, d_n)$. Let A be the adjacent matrix of $\mathcal{G}$; then $D = \mathrm{diag}(A\mathbf{1})$ (where $\mathbf{1}$ is the vector of all ones).

The *Laplacian matrix* of a weighted digraph $\mathcal{G}$ is $L := D - A$. By definition $L\mathbf{1} = 0$; namely each row of L sums to zero. Thus 0 is an eigenvalue of L, with a corresponding eigenvector $\mathbf{1}$.

We distinguish three types of Laplacian matrices depending on their entries. Each type is useful for a set of cooperative control problems introduced in later chapters.

- If A is nonnegative, then L has nonnegative diagonal entries and nonpositive off-diagonal entries. This L is called *standard Laplacian matrix*.
- If A is (arbitrary) real, then L is called *signed Laplacian matrix*.
- If A is complex, then L is called *complex Laplacian matrix*.

Continuing the example in Fig. 1.10, the degree matrix is $D := \text{diag}(d_1, d_2, d_3, d_4, d_5)$, where $d_1 = a_{12}$, $d_2 = a_{21}$, $d = a_{31} + a_{32} + a_{35}$, $d_4 = a_{41} + a_{43} + a_{45}$, and $d_5 = a_{52} + a_{54}$. Thus the Laplacian matrix is

$$L := \begin{bmatrix} d_1 & -a_{12} & 0 & 0 & 0 \\ -a_{21} & d_2 & 0 & 0 & 0 \\ -a_{31} & -a_{32} & d_3 & 0 & -a_{35} \\ -a_{41} & 0 & -a_{43} & d_4 & -a_{45} \\ 0 & -a_{52} & 0 & -a_{54} & d_5 \end{bmatrix}.$$

Since 0 is by definition an eigenvalue of Laplacian matrix L, its *kernel* (i.e. *null space*)[2] is at least one-dimensional. It turns out that the dimensions of the kernel of Laplacian matrices play a central role in characterizing the types of allowable cooperative behaviors.

Remark 1.1 *It is sometimes convenient to define degree matrix and Laplacian matrix with respect to the out-degrees of nodes. Consider a weighted digraph $\mathcal{G}$ and its adjacency matrix A. The* out-degree matrix *of $\mathcal{G}$ is $D^o := \text{diag}(d_1^o, \ldots, d_n^o)$; hence $D^o = \text{diag}(\mathbf{1}^\top A)$. Correspondingly, the* out-degree Laplacian matrix *of $\mathcal{G}$ is $L^o := D^o - A$. By this definition $\mathbf{1}^\top L^o = 0$; namely each column of L^o sums to zero. Thus 0 is again an eigenvalue of L^o, with a corresponding left-eigenvector $\mathbf{1}$.*

1.4 Standard Laplacian Matrices

Let $\mathcal{G}$ be a weighted digraph with n nodes, A the associated adjacency matrix, and $D(= \text{diag}(A\mathbf{1}))$ the degree matrix. In this section we consider that A is nonnegative, and $L = D - A$ the standard Laplacian matrix.

The kernel of L is at least one-dimensional, for L has at least one eigenvalue 0. The following is a graphical condition that characterizes when the kernel of L is exactly one-dimensional (namely the 0 eigenvalue of L is simple). We use $\dim(\cdot)$ to denote the dimension of a vector space.

Theorem 1.7 *Let $\mathcal{G}$ be a weighted digraph with n nodes and L the standard Laplacian matrix. Then $\dim(\ker L) = 1$ if and only if $\mathcal{G}$ contains a spanning tree.*

Note that $\dim(\ker L) = 1$ is equivalent to $\text{rank}(L) = n - 1$. To prove Theorem 1.7, it is useful to first present the following sufficient condition for $\text{rank}(L) = n - 1$.

[2]Kernel of matrix L (viewed as a linear map) is defined as $\ker L := \{v \mid Lv = 0\}$.