Multi-Agent Systems

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3D Formation

- positions cannot be represented by complex numbers
- complex Laplacian cannot be used

Multi-agent system

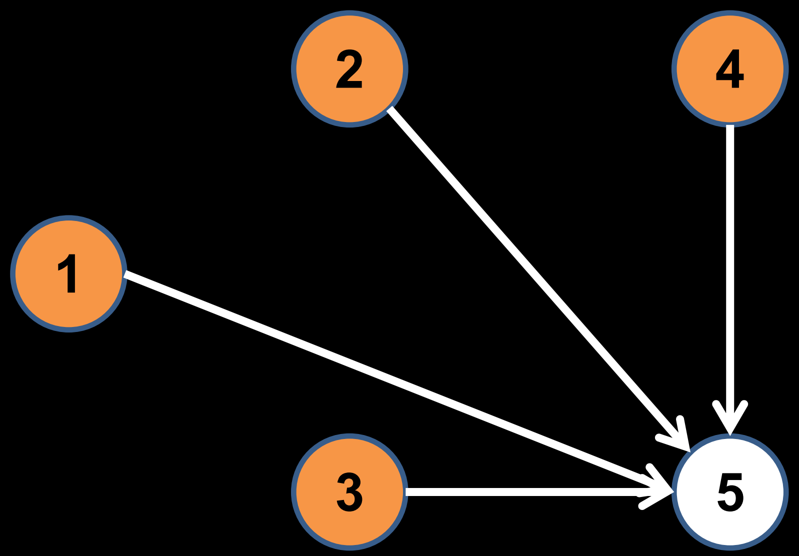
a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node $v_i \in \mathcal{V}$: an agent

edge $(v_j, v_i) \in \mathcal{E}$: agent j sends information to v_i

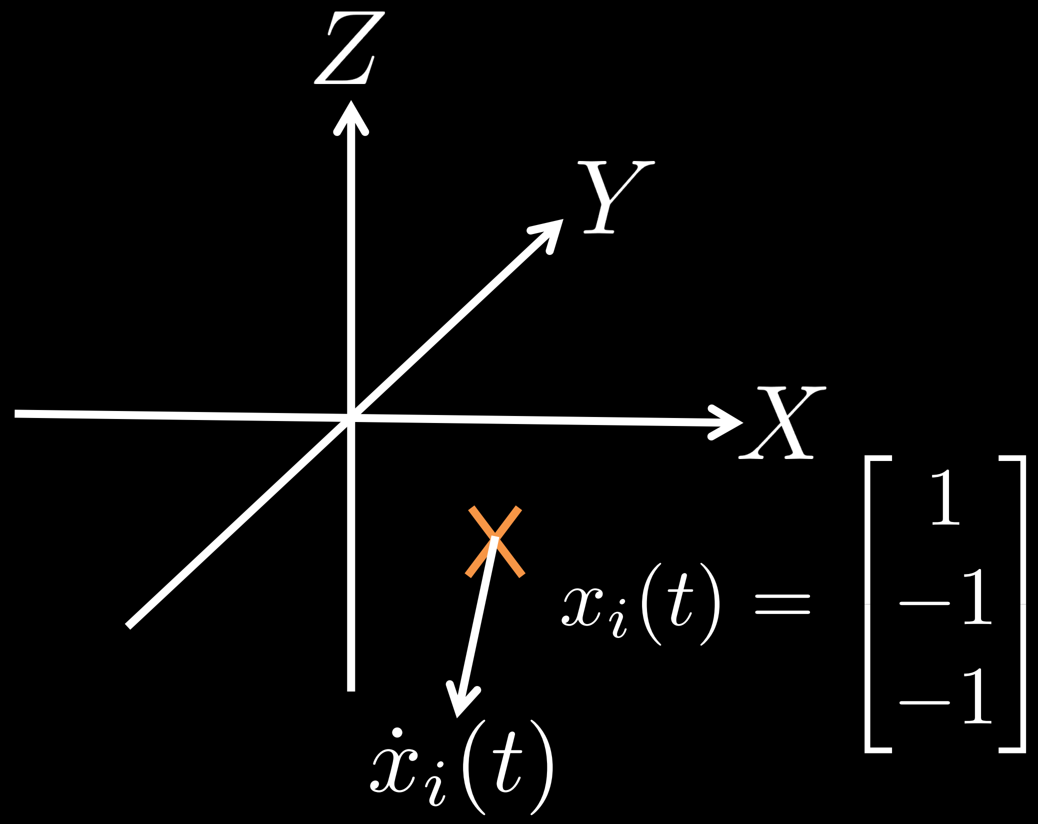
example:



3D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ \ x_i, u_i \in \mathbb{R}^3$$



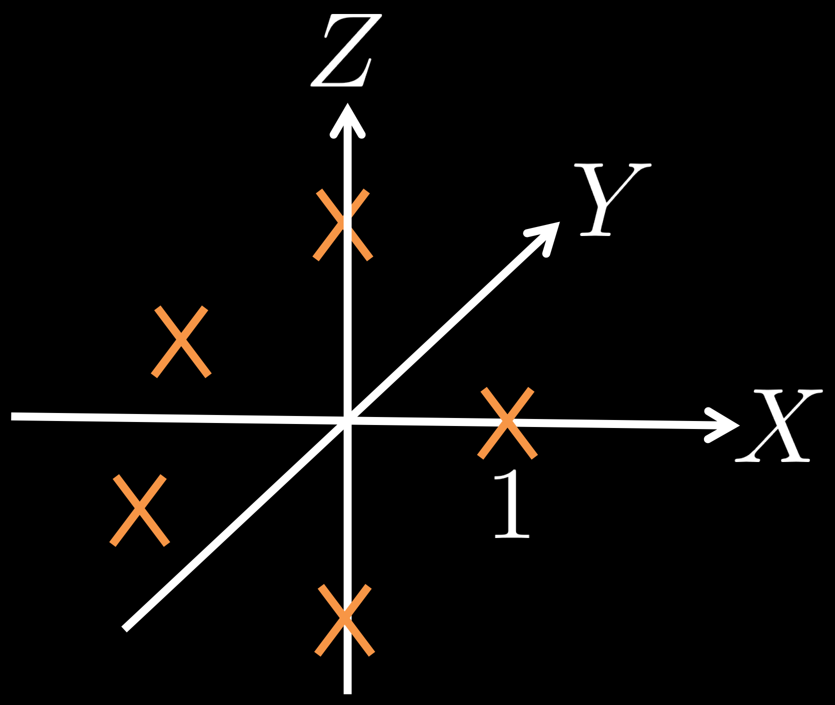
3D formation problem

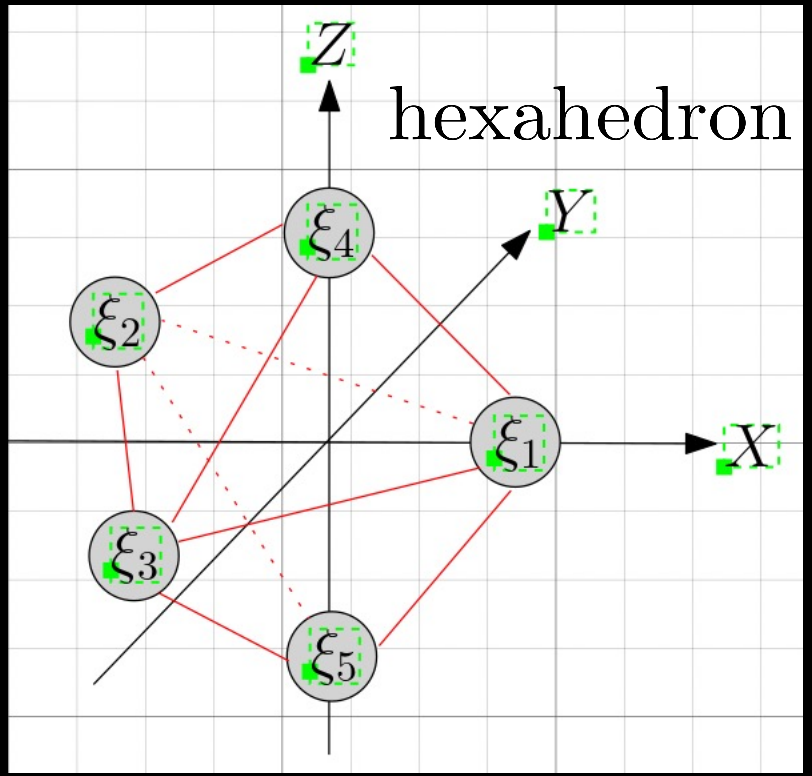
each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration:

$$\xi = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}^\top \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top \begin{bmatrix}0\\0\\1\end{bmatrix}^\top \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \right]^\top$$



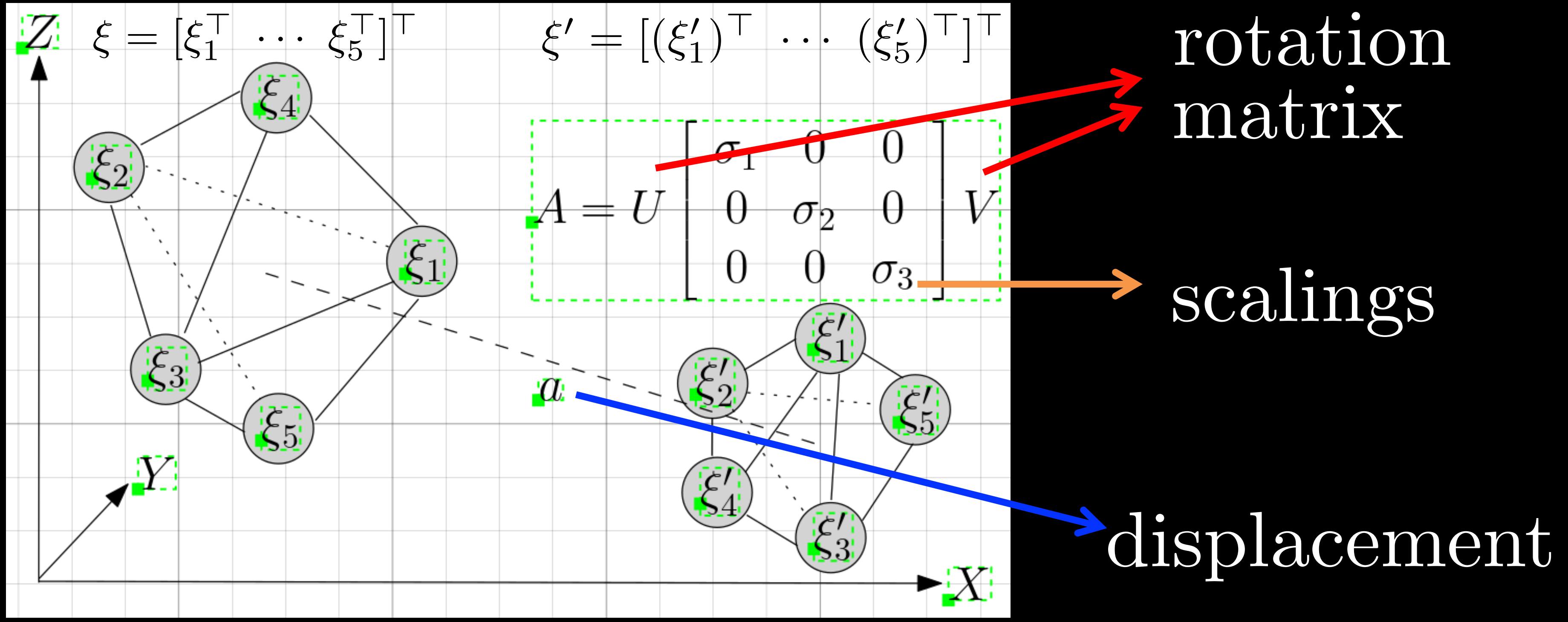


3D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \ \ x_i, u_i \in \mathbb{R}^3$$

3D affine formation:



3D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ x_i, u_i \in \mathbb{R}^3$$

3D affine formation:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} \rightarrow \begin{bmatrix} A\xi_1 + a \\ A\xi_2 + a \\ A\xi_3 + a \\ A\xi_4 + a \\ A\xi_5 + a \end{bmatrix}$$

$$= \begin{bmatrix} A & & & & \\ & A & & & \\ & & A & & \\ & & & A & \\ & & & & A \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + \begin{bmatrix} a \\ a \\ a \\ a \\ a \end{bmatrix}$$

3D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ \ x_i, u_i \in \mathbb{R}^3$$

3D affine formation:

$$\begin{bmatrix} A & & & & \\ & A & & & \\ & & A & & \\ & & & A & \\ & & & & A \end{bmatrix} = I_5 \otimes A$$

$$\begin{bmatrix} a \\ a \\ a \\ a \\ a \end{bmatrix} = \mathbf{1}_5 \otimes a$$

Kronecker product

3D formation problem

each agent v_i updates its state based on

$$\dot{x}_i = u_i,\ x_i, u_i \in \mathbb{R}^3$$

3D affine formation:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} \to (I_5 \otimes A) \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + (\mathbf{1}_5 \otimes a)$$

for n agents:

$$x(t) \to (I_n \otimes A)\xi + \mathbf{1}_n \otimes a$$

3D formation problem

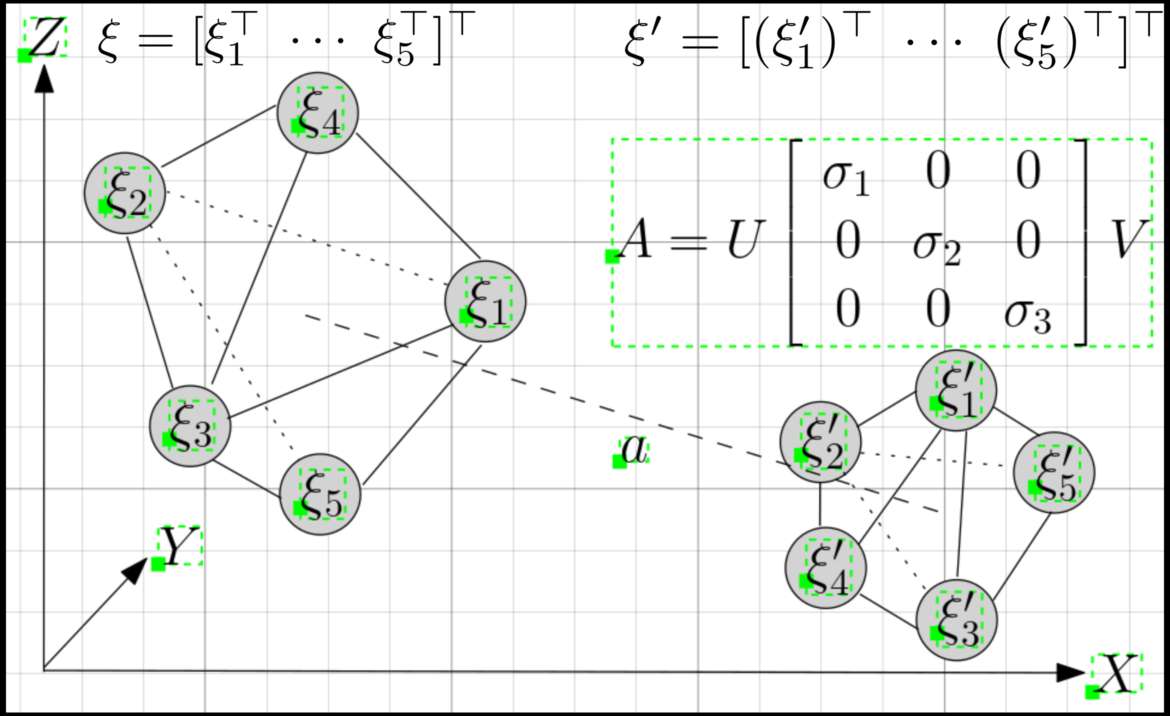
each agent v_i updates its state based on

$$\dot{x}_i = u_i, \ \ x_i, u_i \in \mathbb{R}^3$$

3D affine formation: design input u_i

s.t. $(\forall x_i(0))(\exists A \in \mathbb{R}^{3\times 3}, a \in \mathbb{R}^3)$

$$x(t) \to (I_n \otimes A)\xi + \mathbf{1}_n \otimes a$$



$$U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V$$

singular value decomposition

3D formation problem

singular value decomposition e.g.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2601 & 0.5945 & -0.7608 \\ -0.5074 & -0.7546 & -0.4162 \\ -0.8215 & 0.2778 & 0.4979 \end{bmatrix} \begin{bmatrix} 9.1822 & 0 & 0 \\ 0 & 0.6699 & 0 \\ 0 & 0 & 0.4877 \end{bmatrix}$$

$$\begin{bmatrix} -0.4967 & -0.552 & -0.6698 \\ 0.2933 & -0.8331 & 0.469 \\ 0.8169 & -0.0366 & -0.5757 \end{bmatrix}$$

$$(\text{matlab: } [U, \Sigma, V^\top] = \text{svd}(A))$$

Linear constraint

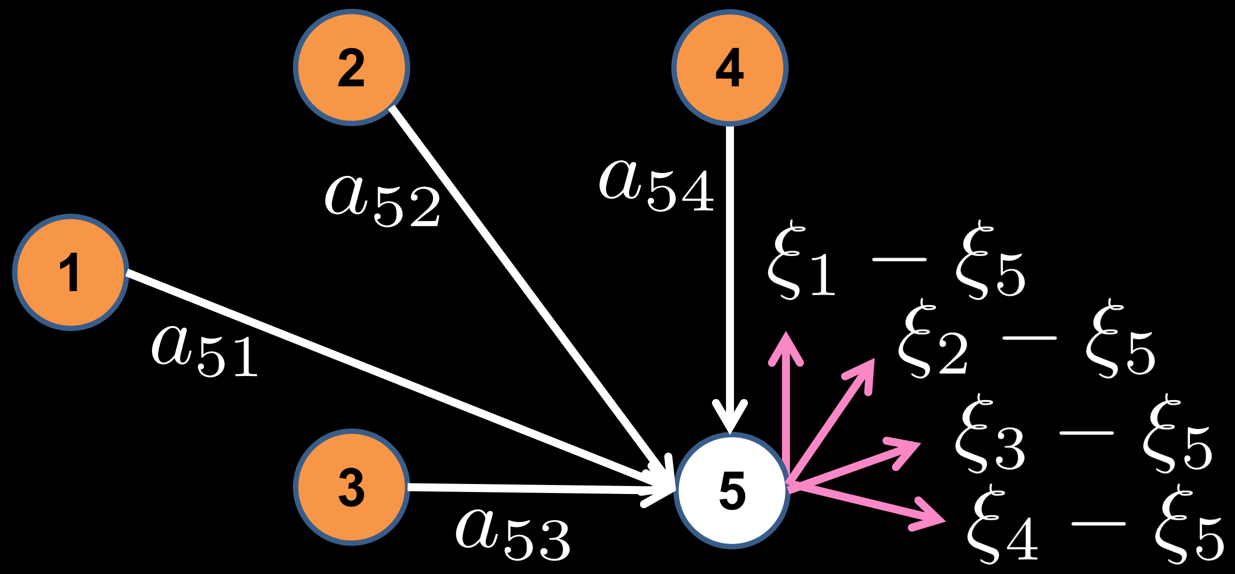
target configuration:

$$\xi = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}^\top \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top \begin{bmatrix}0\\0\\1\end{bmatrix}^\top \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \right]^\top$$

$$\xi_1 - \xi_5 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \xi_2 - \xi_5 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\xi_3 - \xi_5 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \xi_4 - \xi_5 = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

example:



Linear constraint

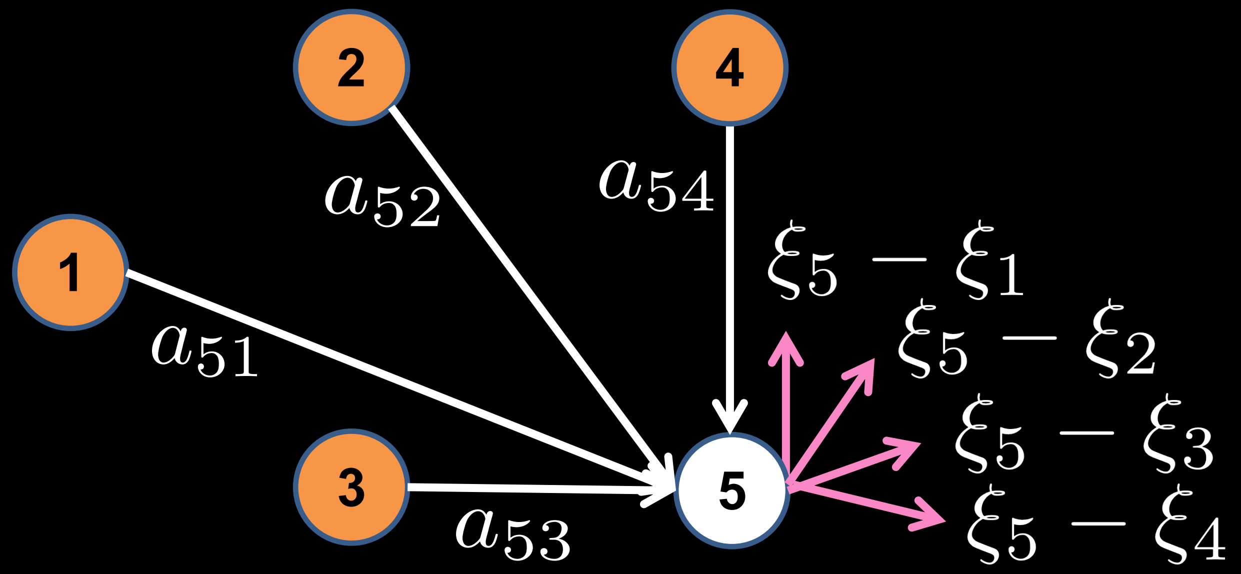
target configuration:

$$\xi = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}^\top \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top \begin{bmatrix}0\\0\\1\end{bmatrix}^\top \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \right]^\top$$

formation constraint:

$$a_{51}(\xi_1 - \xi_5) + a_{52}(\xi_2 - \xi_5) +$$
$$a_{53}(\xi_3 - \xi_5) + a_{54}(\xi_4 - \xi_5) = 0$$

example:



Linear constraint

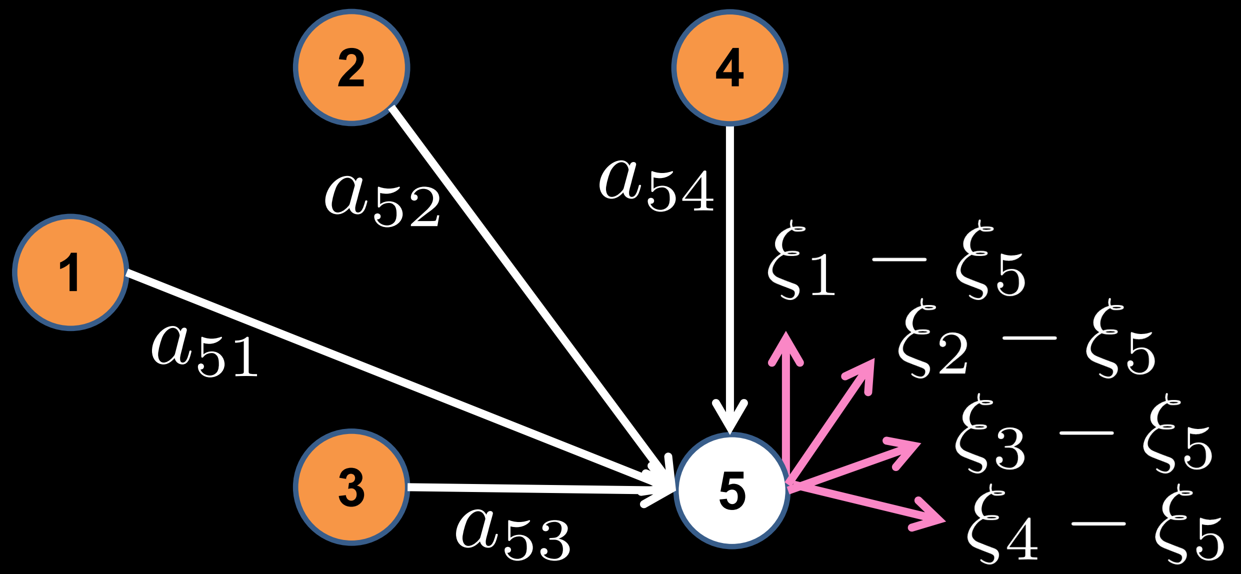
target configuration:

$$\xi = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top & \begin{bmatrix}0\\0\\1\end{bmatrix}^\top & \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \end{bmatrix}^\top$$

formation constraint:

$$a_{51}\begin{bmatrix}1\\0\\1\end{bmatrix} + a_{52}\begin{bmatrix}-1\\1\\1\end{bmatrix} + a_{53}\begin{bmatrix}-1\\-1\\1\end{bmatrix} + a_{54}\begin{bmatrix}0\\0\\2\end{bmatrix} = 0$$

example:



Linear constraint

target configuration:

$$\xi = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^\top \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}^\top \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}^\top \right]^\top$$

formation constraint:

$$a_{51} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_{52} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + a_{53} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + a_{54} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 0$$

e.g. $a_{51} = -1$

$$a_{52} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + a_{53} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + a_{54} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Linear constraint

target configuration:

$$\xi = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top & \begin{bmatrix}0\\0\\1\end{bmatrix}^\top & \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \end{bmatrix}^\top$$

formation constraint:

$$a_{51}\begin{bmatrix}1\\0\\1\end{bmatrix} + a_{52}\begin{bmatrix}-1\\1\\1\end{bmatrix} + a_{53}\begin{bmatrix}-1\\-1\\1\end{bmatrix} + a_{54}\begin{bmatrix}0\\0\\2\end{bmatrix} = 0$$

e.g. $a_{51} = -1$

$$\begin{bmatrix}-1 & -1 & 0\\ 1 & -1 & 0\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}a_{52}\\a_{53}\\a_{54}\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$a_{52} = -0.5$$
$$a_{53} = -0.5$$
$$a_{54} = 1$$

Linear constraint

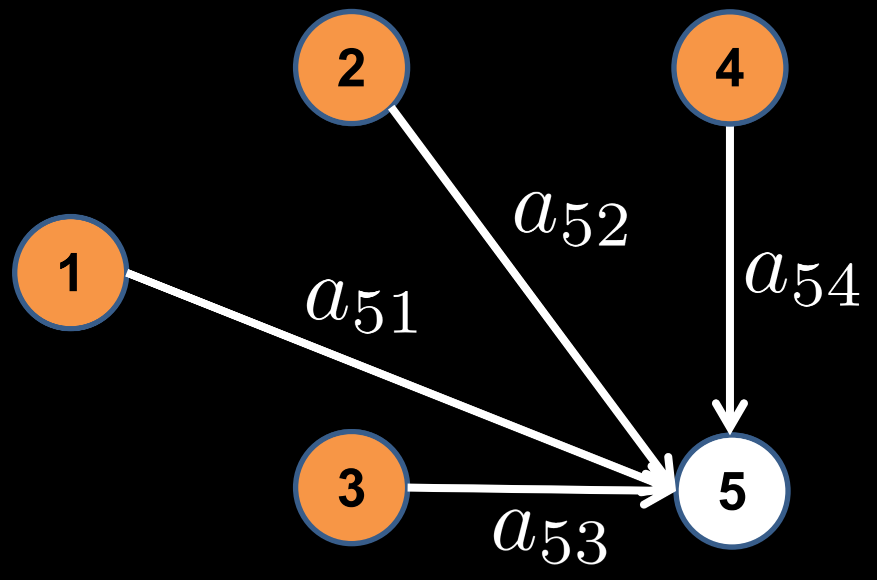
target configuration:

$$\xi = \left[\begin{bmatrix}1\\0\\0\end{bmatrix}^\top \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top \begin{bmatrix}0\\0\\1\end{bmatrix}^\top \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \right]^\top$$

formation constraint:

$$\sum_{j\in\mathcal{N}_5} a_{5j}(\xi_j - \xi_5) = 0$$

example:



Linear constraint

target configuration:

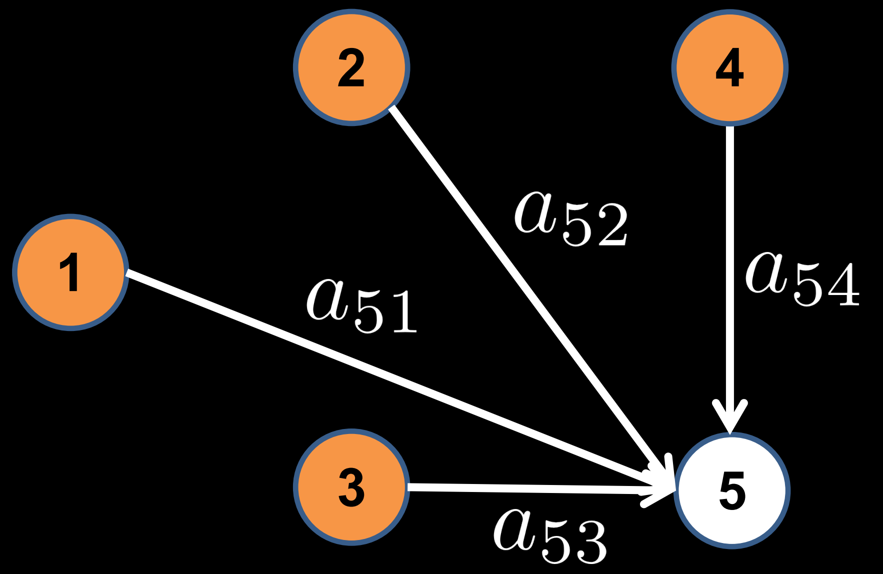
$$\xi = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top & \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}^\top \end{bmatrix}^\top$$

formation constraint:

$$(\forall i = 1, \ldots, n) \sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0$$

$$(L \otimes I_3)\xi = 0$$

example:



Signed Laplacian

target configuration:

$$\xi = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\1\\0\end{bmatrix}^\top & \begin{bmatrix}-1\\-1\\0\end{bmatrix}^\top & \begin{bmatrix}0\\0\\1\end{bmatrix}^\top & \begin{bmatrix}0\\0\\-1\end{bmatrix}^\top \end{bmatrix}^\top$$

signed Laplacian:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

$$(L \otimes I_3)\xi = 0$$

$$L\mathbf{1} = 0$$

$$\Rightarrow \operatorname{rank}(L) \leq 1$$

$$\operatorname{rank}(L) = 1$$

Distributed algorithm

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

distributed algorithm:

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), a_{ij} \in \mathbb{R}$$

relative state information

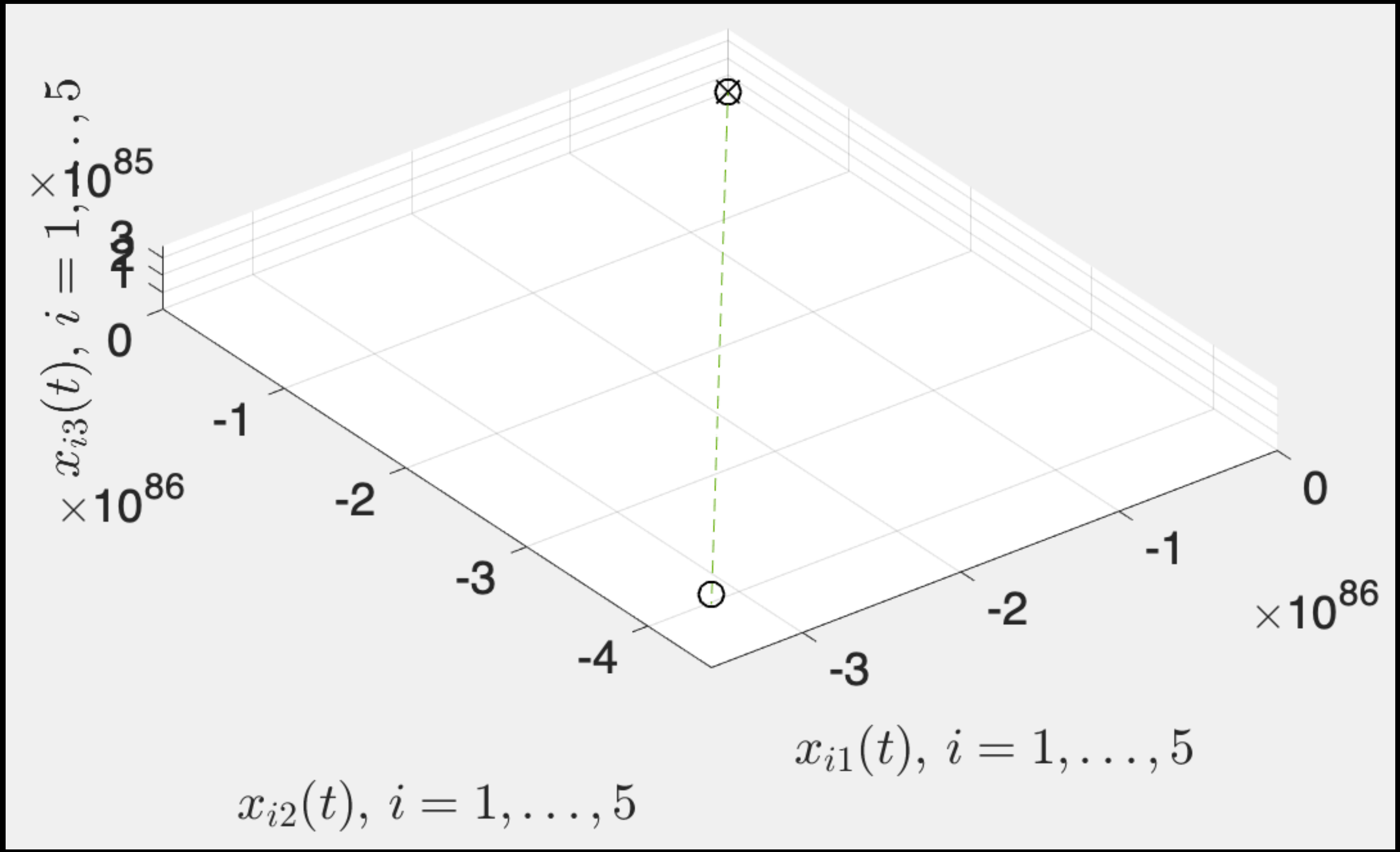
$$\dot{x} = -(L \otimes I_3)x$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

signed Laplacian

Example

simulation: $x_1(0) = [0.88\ 0.52\ 0.94]^\top$
$x_2(0) = [0.64\ 0.96\ 0.24]^\top, x_3(0) = [0.68\ 0.29\ 0.67]^\top$
$x_4(0) = [0.7\ 0.07\ 0.25]^\top, x_5(0) = [0.22\ 0.67\ 0.84]^\top$



Example

$$\dot{x} = -(L \otimes I_3)x$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

eigenvalues of $-L$: $0, 0, 0, 0, 1$

design invertible diagonal matrix E s.t. nonzero eigenvalues of $-EL$ have negative real parts

Distributed algorithm

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

distributed algorithm:

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i} \epsilon_i a_{ij}(x_j - x_i), \; \epsilon_i, a_{ij} \in \mathbb{R}$$

ϵ_i (ensuring stability)

a_{ij} (encoding target configuration)

$$\dot{x} = -(EL \otimes I_3)x, \text{ where } E = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$$

Example

$$\dot{x} = -(EL \otimes I_3)x$$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$L' = -1$

design $E' = \epsilon_5$ s.t. all eigenvalues of $-E'L'$ are stable
equivalently, $E'L'$ has positive real parts
(λ_5)

e.g. $\epsilon_5 = -1$

set $\lambda_5 := \epsilon_5 L' = 1$

Example

$$\dot{x} = -(EL \otimes I_3)x$$

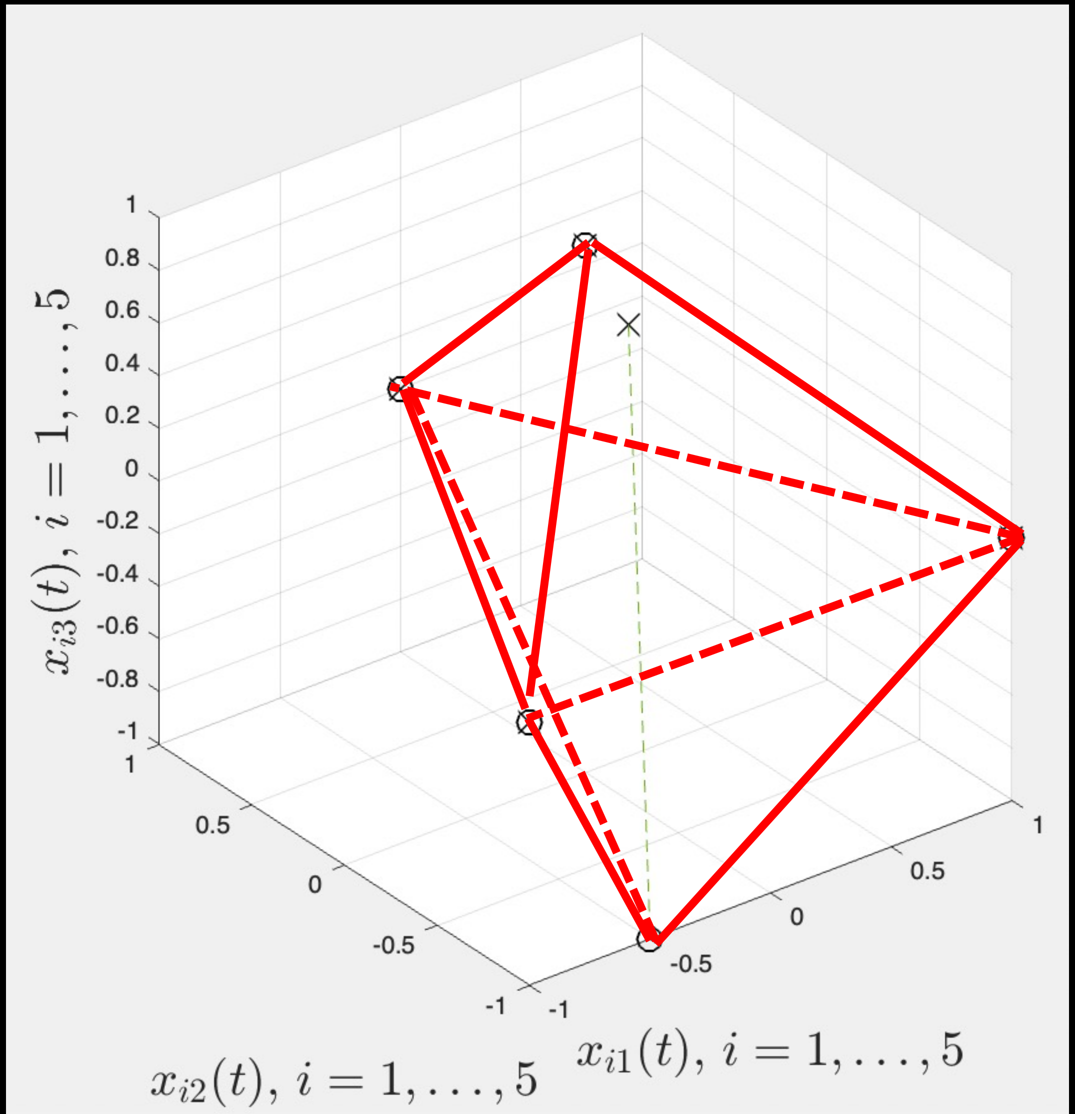
$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

eigenvalues of $-EL$: $0, 0, 0, 0, -1$

Example

simulation: $x_1(0) = [0.88\ 0.52\ 0.94]^\top$
$x_2(0) = [0.64\ 0.96\ 0.24]^\top, x_3(0) = [0.68\ 0.29\ 0.67]^\top$
$x_4(0) = [0.7\ 0.07\ 0.25]^\top, x_5(0) = [0.22\ 0.67\ 0.84]^\top$



Fact

$$\dot{x} = -(EL \otimes I_3)x$$

If $\mathcal{G}$ contains a spanning 4-tree
and ξ *generic*
(no 3 points on the same line,
no 3 lines through the same point)

then E exists s.t. $n-4$ nonzero
eigenvalues of $-EL$ are stable

[Friedland, 1975]

global computation (need L)

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

Problem: design u_i to update x_i

s.t. $(\forall v_i \in \mathcal{V})(\forall x_i(0))(\exists A \in \mathbb{R}^{3\times 3}, a \in \mathbb{R}^3)$

$$\lim_{t\to\infty} x(t) = (I_n \otimes A)\xi + \mathbf{1}_n \otimes a$$

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

affine formation set

$$\mathcal{A}(\xi) = \{\xi' : (\exists A, a)\xi' = (I_n \otimes A)\xi + \mathbf{1}_n \otimes a\}$$

Problem: design u_i to update x_i

$$(\forall x(0) \in \mathbb{R}^{3n})(\exists \xi' \in \mathcal{A}(\xi)) \lim_{t \to \infty} x(t) = \xi'$$

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

Distributed algorithm

$$\dot{x}_i = u_i = \epsilon_i \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \epsilon_i, a_{ij} \in \mathbb{R}$$

where $\sum_{j \in \mathcal{N}_i} a_{ij}(\xi_j - \xi_i) = 0$

based on $x_j(t) - x_i(t)$, $\xi_j - \xi_i$
from neighbor agent(s) $j \in \mathcal{N}_i$

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

Distributed algorithm

$\dot{x} = -(EL \otimes I_3)x$, where $E = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$

$L\mathbf{1} = 0$

signed Laplacian

$(L \otimes I_3)\xi = 0$

$\mathrm{rank}(L) \leq n - 4$ and $\ker(L \otimes I_3) \supseteq \mathcal{A}(\xi) \quad (?)$

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{2n}$

if ξ is generic and $\mathcal{G}$ contains
a spanning 4-tree

then $\text{rank}(L) = n - 4$ and $\ker(L \otimes I_3) = \mathcal{A}(\xi)$

and there exists E s.t. $n - 4$ nonzero
eigenvalues of $-EL$ are stable

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \;\; x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

if ξ is generic and $\mathcal{G}$ contains
a spanning 4-tree

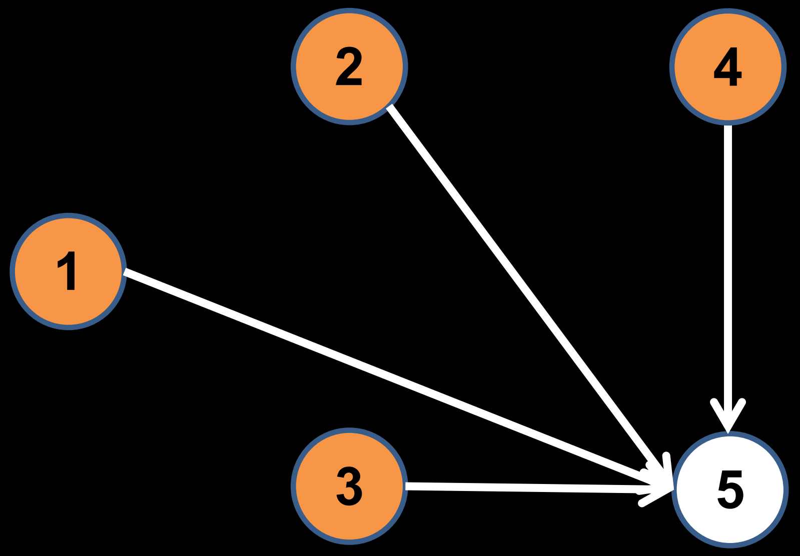
then there exists E s.t. $\dot{x} = -(EL \otimes I_3)x$
solves 3D formation problem

$$(\forall x(0) \in \mathbb{R}^{3n})(\exists \xi' \in \mathcal{A}(\xi)) \lim_{t \to \infty} x(t) = \xi'$$

Example

example:

weighted graph $\mathcal{G}$



spanning 4-tree (?)

generic configuration (?)

$$\xi = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}^\top & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^\top & \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}^\top \end{bmatrix}^\top$$

Example

$$\dot{x} = -(EL \otimes I_3)x$$

signed Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

$$\text{rank}(L) = 1$$

$$\text{so } \ker(L \otimes I_3) = \mathcal{A}(\xi)$$

Example

$$\dot{x} = -(EL \otimes I_3)x$$

signed Laplacian matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & -1 & -1 \end{bmatrix}$$

stabilizing diagonal matrix

$$E = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

s.t. eigenvalues of $-EL$: $0, 0, 0, 0, -1$

note: $\ker(-EL \otimes I_3) = \ker(L \otimes I_3) = \mathcal{A}(\xi)$

Example

$$\dot{x} = -(EL \otimes I_3)x$$

if nonzero eigenvalues of $-EL$ are stable

then $x(t) \rightarrow \underbrace{\ker(-EL \otimes I_3)}_{\mathcal{A}(\xi)}$ as $t \rightarrow \infty$

this means $x(t)$ converges to affine formation of target ξ

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i updates its state based on

$$\dot{x}_i = u_i, \ \ x_i, u_i \in \mathbb{R}^3$$

target configuration $\xi = [\xi_1^\top \cdots \xi_n^\top]^\top \in \mathbb{R}^{3n}$

if ξ is generic and $\mathcal{G}$ contains
a spanning 4-tree

then there exists E s.t. $\dot{x} = -(EL \otimes I_3)x$
solves 3D formation problem

$$(\forall x(0) \in \mathbb{R}^{3n})(\exists \xi' \in \mathcal{A}(\xi)) \lim_{t\to\infty} x(t) = \xi'$$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 4-tree, find a diagonal matrix E s.t. $\dot{x} = -(EL \otimes I_3)x$ solves 3D formation

(i) $\text{rank}(L) = n - 4$

hint: $L\mathbf{1} = 0$, $(L \otimes I_3)\xi = 0$

$$\Rightarrow \text{rank}(L) \leq n - 4$$

$$\text{spanning 4-tree} \Rightarrow \text{rank}(L) \geq n - 4$$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 4-tree, find a diagonal matrix E s.t.

$\dot{x} = -(EL \otimes I_3)x$ solves 3D formation

(ii) $-EL$ has four zero eigenvalues

hint: $\text{rank}(E) = n \Rightarrow \text{rank}(EL) = \text{rank}(L)$

Theorem

Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 4-tree, find a diagonal matrix E s.t. $\dot{x} = -(EL \otimes I_3)x$ solves 3D formation

(iii) $n-4$ nonzero eigenvalues of $-EL$ have negative real parts, i.e. stable

hint: $\mathcal{G}$ contains a spanning 4-tree and ξ generic $\Rightarrow$ E exists s.t. $n-4$ nonzero eigenvalues of $-EL$ are stable

[Friedland, 1975]

Theorem

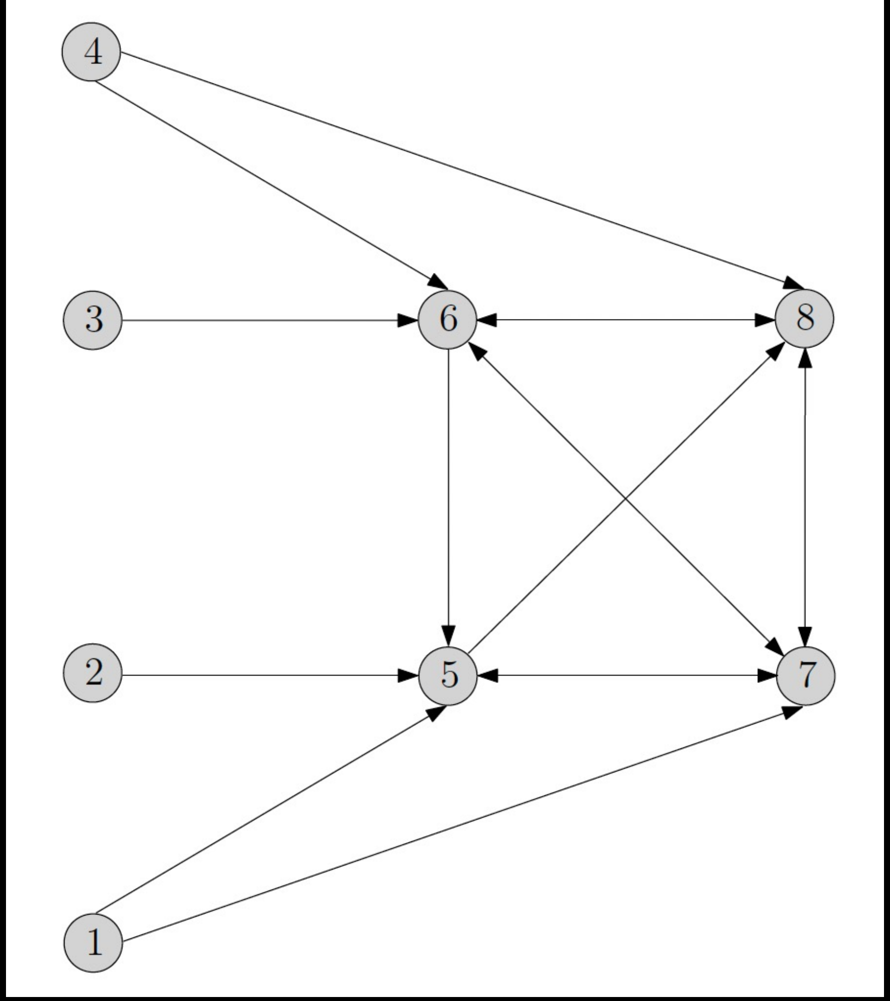
Proof:

if ξ is generic and $\mathcal{G}$ contains a spanning 4-tree, find a diagonal matrix E s.t.

$\dot{x} = -(EL \otimes I_3)x$ solves 3D formation

(iv) $x(t) \to \ker(-EL \otimes I_3) = \mathcal{A}(\xi)$

Example



8 networked agents

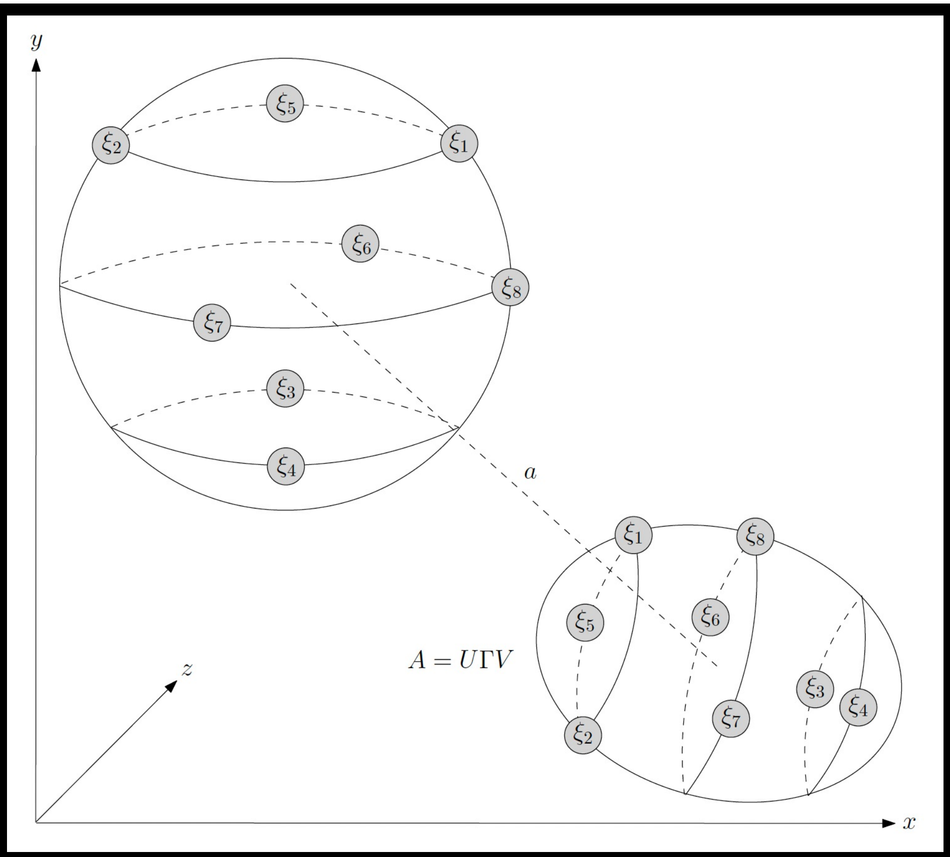
digraph $\mathcal{G}$ contains a spanning 4-tree

Example

generic configuration:

$$\xi_1 = \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_2 = \begin{bmatrix} -\cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -\cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ \cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix},$$

$$\xi_5 = \begin{bmatrix} 0 \\ -\cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix}, \xi_6 = \begin{bmatrix} \cos\frac{\pi}{3} \\ -\sin\frac{\pi}{3} \\ 0 \end{bmatrix}, \xi_7 = \begin{bmatrix} -\cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \\ 0 \end{bmatrix}, \xi_8 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$



Example

8 networked agents