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On algebraic connectivity of directed scale-free networks

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Abstract

In this paper, we study the algebraic connectivity of directed complex networks with scale-free property. Algebraic connectivity of a directed graph is the eigenvalue of its Laplacian matrix whose real part is the second smallest. This is known as an important measure for the diffusion speed of many diffusion processes over networks (e.g. consensus, information spreading, epidemics). We propose an algorithm, extending that of Barabasi and Albert, to generate directed scale-free networks, and show by simulations the relations between algebraic connectivity and network size, exponents of in/out-degree distributions, and minimum in/out degrees. The results are moreover compared to directed small-world networks, and demonstrated on a specific diffusion process, reaching consensus.

1. Introduction

Recently the scale of many real networks has grown larger and their topologies become more complex. In response, many network models have been proposed to analyze the topological property of real networks [1–9]. Watts and Strogatz [3] presented a network model that generates a *small-world* network, from a regular network by rewiring some edges with a fixed probabilities. This network has a property that any two nodes in the network can be linked within a few steps even if the network is large, while nodes are still highly clustered. After this model, Barabasi and Albert [5] introduced the model of *scale-free* network,

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in which there are many nodes with low degrees, while a few nodes called *hubs* have very high degrees. In this model, the network gradually grows and an added node is linked to an existing node based on *preferential attachment*: the added node is more probable to connect to an existing node of high degree. It was shown in [5] that the degree distribution of the network generated from this model follows a power law with an exponent.

Analyzing the spectral properties of complex networks has attracted much attention, in particular the second smallest eigenvalue of the associated *Laplacian* matrices [7,10–16]. This special eigenvalue is referred to as the *algebraic connectivity* of networks [17–20], and is known as an important measure for the diffusion speed of many diffusion processes over networks (e.g. consensus, synchronization, information/innovation spreading, epidemics) [15,19]. In [10] it was demonstrated that undirected small-world networks have much higher algebraic connectivity than regular networks. A similar observation was reported in [14] for directed small-world networks. On the other hand, algebraic connectivity of undirected scale-free networks, specifically on the correlations among the exponent of degree distribution, the minimum degree of the network, and the algebraic connectivity. It was found that algebraic connectivity increases as the exponent or the minimum degree increases.

In many real, scale-free networks such as social networking service (SNS) and World Wide Web (WWW), however, the edges may not be bidirectional. For example, in Twitter, we can follow some (popular) people, but they do not necessarily follow us; in WWW, a webpage can have links to some (well-known) pages, which may not have links back to that webpage. These have motivated us to study the algebraic connectivity of *directed* scale-free networks.

In this paper, we first propose a new algorithm that provably generates directed scale-free networks. This algorithm is a natural extension of the Barabasi and Albert (BA) model [5] from undirected to directed networks: starting from an initial directed network, one node is added at a time with m_{in} *in-edges* from, and m_{out} *out-edges* to, the existing nodes by preferential attachment. Thus when the algorithm stops and a network is generated, it is easy to calculate the number of nodes and in/out-edges of the network. Moreover, we show that the exponents of in-degree and out-degree distributions are determined only by m_{in} and m_{out}, respectively.

Using this algorithm, we investigate by simulations the impacts of *structural* properties of directed scale-free networks (size, exponents of in/out-degree distributions, minimum in/out-degrees) on the algebraic connectivity. Specifically, it is found that algebraic connectivity (i) stays roughly the same in spite of increase of network size (here measured by number of nodes); (ii) increases (resp. decreases) as the exponent of in-degree (resp. out-degree) distribution increases; (iii) increases (resp. decreases) as the minimum in-degree (resp. out-degree) increases. Moreover, we compare directed scale-free networks with directed small-world networks, and demonstrate that for the same number of nodes and comparable number of edges, directed scale-free networks have larger algebraic connectivity. This means that the speed of diffusion over directed scale-free networks is faster than that over directed small-world networks, as illustrated by studying the consensus problem (a representative example of diffusion processes over networks).

We note that several previous works also reported algorithms for generating directed scale-free networks [21–24]. However, the algebraic connectivity of the generated networks was not studied. Moreover, by the algorithms reported in [21–23], the generated scale-free networks have power-law only for in-degree distribution; by contrast, our algorithm generates directed scale-free networks which have power-law for both in-degree and out-degree distributions and the exponents of in/out-degree distributions may be specified. On the other hand, the algorithm in [24] can construct directed scale-free networks which have power-law in and out-degree distributions. However, at each iteration of the algorithm in [24], it may happen (with probability) that a new node is not added but a new edge is added (to a pair of existing nodes); thus the number of nodes and the number of edges are not deterministic for each generated network (they are indeed random variables). By contrast, our algorithm, extending the BA model to the directed case, generates scale-free networks with easily calculatable (deterministic) numbers of nodes/edges; this is particularly useful when we compare the algebraic connectivity of these networks to directed small-world ones with the same number of nodes and comparable number of edges.

The outline of the rest of this paper is as follows. In Section 2, we introduce preliminaries on algebraic graph theory and directed scale-free networks. In Section 3, we present an algorithm and prove that it generates directed scale-free networks. In Section 4, we provide simulation results on the relations between the structural properties of directed scale-free networks and algebraic connectivity. Finally, our conclusions are stated in Section 5.

2. Preliminaries

Consider a directed graph $\mathcal{D} = (V, E)$ with N nodes. Here $V = \{1, 2, \ldots, N\}$ and $E \subseteq V \times V$ represent the set of nodes and the set of edges of $\mathcal{D}$, respectively. Then the adjacency matrix $A(\mathcal{D})$ is defined as

$$A_{ij}(\mathcal{D}) = \begin{cases} 1 & (\text{if } (i, j) \in E) \\ 0 & (\text{otherwise}). \end{cases}$$

Note that $(i, j) \in E$ need not imply $(j, i) \in E$; hence $A(\mathcal{D})$ is asymmetric in general. For node i, an edge $(j, i) \in E$ is called an *in-edge* while $(i, j) \in E$ an *out-edge*. Let $k_{i,\, in}$ be the *in-degree* of node i, which is the number of in-edges to node i. Then the degree matrix $D(\mathcal{D})$, which is a diagonal matrix consisting of the in-degree of each node, is given by

$$D(\mathcal{D}) := \mathrm{diag}(k_{1,in},\ k_{2,in},\ \ldots, k_{N,in}).$$

The Laplacian matrix associated with the directed graph $\mathcal{D}$, denoted as $L(\mathcal{D})$, is defined by

$$L(\mathcal{D}) := D(\mathcal{D}) - A(\mathcal{D}).$$

Generally speaking, in a directed graph, the adjacency matrix $A(\mathcal{D})$ is asymmetric. Thus the Laplacian matrix $L(\mathcal{D})$ is also asymmetric, and the eigenvalues of $L(\mathcal{D})$ are generally complex. Of particular importance is the eigenvalue whose real part is the second smallest; we refer to the real part of this eigenvalue as the *algebraic connectivity*.[1] It is well-known that the algebraic connectivity determines the diffusion speed of many diffusion processes over networks.[2]

Moreover, a digraph $\mathcal{D}$ contains a *spanning tree* if it has a node that can reach to all other nodes (via directed paths). It is known that $\mathcal{D}$ contains a spanning tree if and only if its Laplacian matrix $L(\mathcal{D})$ has a simple eigenvalue 0.

[1] Algebraic connectivity is originally defined for undirected graphs and refers to the second smallest eigenvalue of the corresponding Laplacian matrix [17].

[2] We note that a distinct definition of algebraic connectivity for directed graphs is reported in [25], which is shown to provide a lower bound on the amount of coupling needed to synchronize an array of chaotic systems. However, the definition in [25] seems unrelated to the speed of diffusions over networks (the subject of study in this paper).

Next, we introduce the *scale-free* property, which is found to be a common feature in many real networks [5]. This property means roughly that many nodes are connected with only a handful of other nodes, while some (hub) nodes with a large number of nodes. Scale-free property can be represented in a mathematical way. Let k_{in} (resp. k_{out}) be the in-degree (resp. out-degree) of a node, namely the number of in-edges (resp. out-edges) of that node. Also let $P(k_{in})$, $P(k_{out})$ be the in-degree distribution and the out-degree distribution, respectively; these are the ratios of the number of nodes with in-degree k_{in} or out-degree k_{out} with respect to the total number of nodes in the network. The scale-free property of directed networks refers to that $P(k_{in})$, $P(k_{out})$ follow the power laws [26]:

$$P(k_{in}) \sim {k_{in}}^{-\gamma_{in}},$$
$$P(k_{out}) \sim {k_{out}}^{-\gamma_{out}},$$

where $\sim$ means "proportional to" and γ_{in}, γ_{out} are called the exponents of the in-degree distribution and the out-degree distribution, respectively. As an example, the in/out-degree distributions of WWW follow power laws with $\gamma_{in} \simeq 2.1$, $\gamma_{out} \simeq 2.7$ [26].

In [5] Barabasi and Albert introduced an algorithm to generate undirected scale-free networks. This algorithm has two essential ingredients: "growth" and "preferential attachment". First, the network grows by adding one new node at each iteration. Second, the probability that the new node is connected to an existing node is proportional to the latter's degree. It was shown [5] that the degree distribution of undirected scale-free networks generated by the BA Algorithm follows a power law.

In this paper we study directed scale-free networks and their algebraic connectivity. For this, we shall design an algorithm to generate directed scale-free networks, by extending the BA Algorithm but maintaining the two main ingredients – "growth" and "preferential attachment".

3. Algorithm for generating directed scale-free networks

First, we present an algorithm to generate directed scale-free (DSF) networks, by extending the BA Algorithm.

Algorithm DSF:

1. Initially let $\mathcal{D}_0$ be a directed graph with $m_0(>1)$ nodes that contains a spanning tree.
2. At each iteration t (≥ 1), add a new node with $m_{in} \in [1, m_0]$ in-edges from and $m_{out} \in [1, m_0]$ out-edges to the existing nodes. The probability $\Pi_{i,\, in}$ (resp. $\Pi_{i,\, out}$) that an existing node i with in-degree $k_{i,\, in}$ (resp. out-degree $k_{i,\, out}$) establishes an in-edge from (resp. out-edge to) the existing node is

$$\Pi_{i,in} = \frac{k_{i,in}}{\sum_j k_{j,in}}, \tag{1}$$

$$\text{resp.} \quad \Pi_{i,out} = \frac{k_{i,out}}{\sum_j k_{j,out}}. \tag{2}$$

The above summations are over all the existing nodes. No self-loop edges or multiple edges are allowed.

3. If $t = N - m_0 - 1$, stop. Otherwise advance t to $t + 1$ and go to Step 2).

When the DSF Algorithm stops, the generated network has N nodes. Let K_0 be the number of edges of the initial network $\mathcal{D}_0$. Then the number of edges of the generated network, denoted by K_{SF}, is

$$K_{SF} := K_0 + (m_{in} + m_{out})(N - m_0 - 1).$$

Since K_0 and m_0 are typically small constants, for large N we can ignore them and write

$$K_{SF} \simeq (m_{in} + m_{out})N. \tag{3}$$

In Step 2) of the DSF Algorithm, the network grows with one new node at each iteration, and the probabilities $\Pi_{i,\, in}$, $\Pi_{i,\, out}$ in Eqs. (1), (2) mean *preferential attachment*: the higher in-degree (resp. out-degree) an existing node has, the more likely it establishes an in-edge from (resp. out-edge to) the newly added node. Although the DSF Algorithm has twice the number of probability calculations of the BA Algorithm, for the (asymptotic) computational complexity in terms of the number N of nodes, our DSF Algorithm and the BA Algorithm are the same: $O(N^2)$.

The growth and preferential attachment features lead to that the network generated by the DSF Algorithm has scale-free property, as asserted by the following theorem.

Theorem 1. *The network generated by the DSF Algorithm has scale-free property, i.e.*

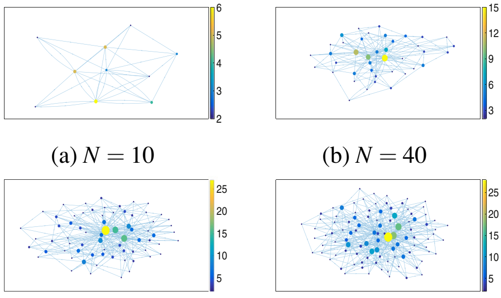
$$P(k_{in}) \sim {k_{in}}^{-\gamma_{in}}$$
$$P(k_{out}) \sim {k_{out}}^{-\gamma_{out}}$$

where $\gamma_{in} = 2 + \frac{m_{in}}{m_{out}}$, $\gamma_{out} = 2 + \frac{m_{out}}{m_{in}}$. *Moreover, the generated network contains a spanning tree.*

Theorem 1 asserts that the directed networks generated by the DSF Algorithm have power-law distributions for both in-degree and out-degree, with exponents γ_{in}, γ_{out} determined solely by the number m_{in} of in-edges and the number m_{out} of out-edges of the newly added nodes. This is in contrast with the scale-free networks generated by the BA Algorithm, whose edges are undirected and there is only a single power-law degree distribution with a single exponent.

In Fig. 1, a directed scale-free network generated by the DSF Algorithm is displayed. The color of the circles represents the value of the nodes' in-degree approaches to yellow as the corresponding in-degree becomes higher. Thus a handful of nodes close to the center are hub nodes with high in-degrees. Note that for directed scale-free networks, the hubs with high in-degrees need not also have high out-degrees. For the network in Fig. 1(d), we display its in-degree and out-degree distributions in Fig. 2. This figure indicates that in the network, many nodes have low in/out-degrees, while a few hub nodes with very hight in/out-degrees exist.

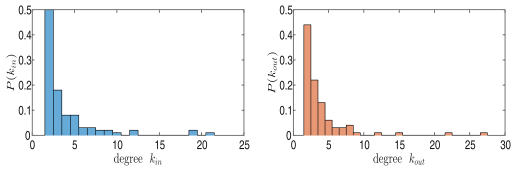
Note that in [24], an algorithm was designed and proved to also generate directed scale-free networks. It is, however, difficult to calculate the number of nodes and the number of edges for the generated networks, because at each iteration of the algorithm it may happen (with probability) that a new node is not added but a new edge is added (to a pair of existing nodes). By contrast, the scale-free networks generated by our designed DSF Algorithm have N nodes and edge number as in Eq. (3). This is useful for our study on the algebraic connectivity of these networks, particularly when making comparisons with other networks with the same number of nodes and comparable number of edges.



(a) $N = 10$ (b) $N = 40$

(c) $N = 70$ (d) $N = 100$

Fig. 1. The process of growth in a directed scale-free network generated by the DSF Algorithm with parameters $m_0 = 2, m_{in} = m_{out} = 2$. The color of each circle represents the in-degree of each node and approaches to yellow as the corresponding in-degree becomes higher.



(a) in-degree (b) out-degree

Fig. 2. In/out-degree distributions of the directed scale-free network in Fig. 1(d).

Now we provide the proof of Theorem 1.

Proof of Theorem 1: First, we show that the DSF Algorithm generates a directed scale-free network. In Step 1), there are initially m_0 nodes in the network, and in Step 2) a new node is added to the network at each iteration. Let node i be the node that is newly added at iteration t_i, and we focus on how its in-degree and out-degree change with respect to $t(\geq t_i)$. Let $k_{i,\,in}(t)$ and $k_{i,\,out}(t)$ be in-degree and out-degree of node i at iteration $t(\geq t_i)$, respectively. Since the newly added node has m_{in} in-edges and m_{out} out-edges, initially we have

$$\begin{aligned} k_{i,in}(t_i) &= m_{in}, \\ k_{i,out}(t_i) &= m_{out}. \end{aligned} \tag{4}$$

At each iteration $t > t_i$, a new node is added to the network and establishes an in-edge from m_{in} distinct nodes and an out-edge to m_{out} distinct nodes. The probability $\Pi_{i,\,in}$ that the new node establishes an edge from node i and the probability $\Pi_{i,\,out}$ that the new node establishes an edge to node i are expressed in Eqs. (1) and (2), respectively. Hence, at each step the expectation of the increase of the in-degree (resp. out-degree) of node i is $m_{out}\Pi_{i,\,in}$ (resp. $m_{in}\Pi_{i,\,out}$). When t is large, i.e. $t - t_i \gg 0$, one may regard t as a continuous variable. With this approximation, the temporal variations of $k_{i,\,in}$ and $k_{i,\,out}$ are represented as

$$\frac{dk_{i,in}}{dt} = m_{out}\Pi_{i,in} = \frac{m_{out}k_{i,in}}{\sum_j k_{j,in}}, \tag{5}$$

$$\frac{dk_{i,out}}{dt} = m_{in}\Pi_{i,out} = \frac{m_{in}k_{i,out}}{\sum_j k_{j,out}}. \tag{6}$$

In the following we focus on the derivation of the in-degree distribution based on Eq. (5); the out-degree distribution based on Eq. (6) is analogous.

The denominator of the right side of Eq. (5) stands for the summation of the in-degrees of all nodes in the network at t. Since at each iteration, the network is added with $m_{in} + m_{out}$ directed edges, we have

$$\sum_j k_{j,in} = K_0 + (m_{in} + m_{out})t.$$

For $t \gg t_i$, the constant K_0 can be ignored and we obtain $\sum_j k_{j,in} = (m_{in} + m_{out})t$. Hence the probability $\Pi_{i,in} = \frac{k_{i,in}}{(m_{in} + m_{out})t}$ and Eq. (5) becomes

$$\frac{dk_{i,in}}{dt} = \frac{m_{out} k_{i,in}}{(m_{in} + m_{out})t}. \tag{7}$$

By solving this differential equation, we have

$$k_{i,in}(t) = At^{\frac{m_{out}}{m_{in}+m_{out}}},$$

where A denotes an integration constant. Using the initial condition Eq. (4), we obtain

$$A = \frac{m_{in}}{t_i^{\frac{m_{out}}{m_{in}+m_{out}}}}.$$

Hence the solution of Eq. (7) is

$$k_{i,in}(t) = m_{in}\left(\frac{t}{t_i}\right)^{\frac{m_{out}}{m_{in}+m_{out}}}.$$

By fixing $k_{i,\,in}(t) =: k_{in}$ and replacing t_i by $t_{k_{in}}$, we have

$$t_{k_{in}} = \left(\frac{m_{in}}{k_{in}}\right)^{1+\frac{m_{in}}{m_{out}}} t.$$

This equation represents the time when the node with in-degree k_{in} at t is added to the network. Let $N_{<k_{in}}$ be the number of nodes whose in-degrees are lower than k_{in} at time t; then this number is equal to the number of nodes which is added after the time $t_{k_{in}}$ and is represented as

$$N_{<k_{in}} = t - \left(\frac{m_{in}}{k_{in}}\right)^{1+\frac{m_{in}}{m_{out}}} t.$$

On the other hand, let $P(k'_{in})$ be the in-degree distribution; then $P(k'_{in})$ represents the ratio of nodes with in-degree k_{in}. Thus $N_{<k_{in}}$ also has the form

$$N_{<k_{in}} = N(t)\int_{m_{in}}^{k_{in}} P(k'_{in})dk'_{in},$$

where $N(t)$ denotes the number of nodes. If $t \gg m_0$, we can ignore m_0, so we have $N(t) = m_0 + t \approx t$. Thus we obtain

$$t\int_{m_{in}}^{k_{in}} P(k'_{in})dk'_{in} = t - \left(\frac{m_{in}}{k_{in}}\right)^{1+\frac{m_{in}}{m_{out}}} t.$$

Dividing both sides by t, we have

$$\int_{m_{in}}^{k_{in}} P(k'_{in})dk'_{in} = 1 - \left(\frac{m_{in}}{k_{in}}\right)^{1+\frac{m_{in}}{m_{out}}}. \tag{8}$$

Thus we can obtain the power distribution by differentiating Eq. (8) with respect to k_{in}:

$$\begin{aligned} P(k_{in}) &\sim \left(1+\frac{m_{in}}{m_{out}}\right) m_{in}^{(1+\frac{m_{in}}{m_{out}})} k_{in}^{-\left(2+\frac{m_{in}}{m_{out}}\right)} \\ &\sim k_{in}^{-\gamma_{in}} \end{aligned} \tag{9}$$

where the exponent $\gamma_{in} = 2 + \frac{m_{in}}{m_{out}}$. In a similar fashion, we derive that the out-degree distribution follows the power law:

$$\begin{aligned} P(k_{out}) &\sim \left(1+\frac{m_{out}}{m_{in}}\right) m_{out}^{(1+\frac{m_{out}}{m_{in}})} k_{out}^{-\left(2+\frac{m_{out}}{m_{in}}\right)} \\ &\sim k_{out}^{-\gamma_{out}} \end{aligned} \tag{10}$$

where the exponent $\gamma_{out} = 2 + \frac{m_{out}}{m_{in}}$. Therefore, it follows that the generated networks have the scale-free property.

It is left to show that the generated network contains a spanning tree. To prove this, we use mathematical induction. By the setup of Step 1) in the algorithm, the initial network $\mathcal{D}_0$ contains a spanning tree. We assume that the network $\mathcal{D}_t$ contains a spanning tree S at iteration $t(\geq 1)$. At iteration $t+1$, a new node establishes m_{in} in-edges from the existing nodes. Thus the new node is reachable from the root of S, and therefore $\mathcal{D}_{t+1}$ also contains a spanning tree. By induction we conclude that generated network contains a spanning tree. □

4. Algebraic connectivity of directed scale-free networks

In this section, we show simulation results on the algebraic connectivity of the directed scale-free networks generated by the DSF Algorithm in Section 3.

4.1. Topological impacts on algebraic connectivity

We illustrate the impacts of topological properties of directed scale-free networks on the algebraic connectivity. We shall focus on three factors: size, exponents of in/out-degree distributions, minimum in/out-degree.

First (size), a ring graph of $m_0 = 4$ nodes is set as the initial network and let $m_{in} = m_{out} = 3$. Vary N from 100 to 1000 and compute the corresponding algebraic connectivity. This investigation is important because growth is one of the two main features of scale-free networks.

In Fig. 3 (with parameters specified in Table 1) each plotted point is an average of 100 simulation runs. Observe that algebraic connectivity stays roughly the same as N increases. This means diffusion rate does not drop as the network expands, which makes directed scale-free networks an ideal model for scalable (fast) diffusion.

In a very special case, directed scale-free networks generated by the DSF Algorithm has a constant algebraic connectivity.

Table 1
Parameter settings for Fig. 3.

γ_{in}	γ_{out}	m_{in}, m_{out}
2.5	4	$m_{in}=2, m_{out}=4$
3	3	$m_{in}=m_{out}=3$
4	2.5	$m_{in}=4, m_{out}=2$

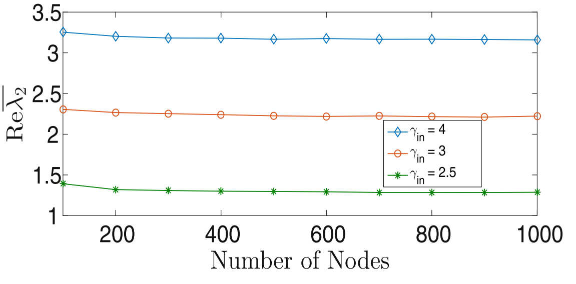


Fig. 3. The impacts of network size and exponents of in/out-degree distributions on algebraic connectivity $\mathrm{Re}\overline{\lambda_2}$ (averaged over 100 simulation runs).

Theorem 2. *In the DSF Algorithm, assume that $m_{in}=m, m_{out}=0$. Then the algebraic connectivity of generated networks $\mathcal{D}$ is constant and determined by*

$$\mathrm{Re}\lambda_2(L(\mathcal{D}))=\min\{\mathrm{Re}\lambda_2(L(\mathcal{D}_0),m\}.$$

Proof. If $m_{in}=m, m_{out}=0$, a newly added node will give no edges to the existing nodes but only receive m edges from them. Thus the Laplacian matrix of network $\mathcal{D}$ is represented as

$$L(\mathcal{D})=\begin{pmatrix} L(\mathcal{D}_0) & O \\ * & mI \end{pmatrix},$$

where $L(\mathcal{D}_0)\in\mathbb{R}^{m_0\times m_0}$ is the graph Laplacian matrix of the initial network $\mathcal{D}_0$, $O\in\mathbb{R}^{(m_0)\times(N-m_0)}$ the zero matrix, $*\in\mathbb{R}^{(N-m_0)\times m_0}$ the matrix consisting of 0 or -1, and $I\in\mathbb{R}^{(N-m_0)\times(N-m_0)}$ the identity matrix. Hence the eigenvalues of $L(\mathcal{D})$ consist of the eigenvalues of $L(\mathcal{D}_0)$ and mI. Since mI is a diagonal matrix, all of its eigenvalues are equal to m. Therefore, the algebraic connectivity of $\mathcal{D}$ is determined the smaller value of $\mathrm{Re}\lambda_2(L(\mathcal{D}_0))$ and m. □

We note that setting $m_{in}=m$, $m_{out}=0$ results in $\gamma_{in}=\infty$, $\gamma_{out}=2$ (by Theorem 1). We consider this choice of m_{in}, m_{out} only in Theorem 2, with the purpose to show that this special choice leads to constant algebraic connectivity of the generated networks.

Theorem 2 may be generalized to the case where the number $m_{in}(t)$ of in-edges of the new node added at iteration t is in the range $[1, m_0+t-1]$, i.e. $1\le m_{in}(t)\le m_0+t-1$.

Theorem 3. *In the DSF algorithm, assume that $m_{in}(t)$ is in the range $[1, m_0+t-1]$, $m_{out}=0$. Then the algebraic connectivity of generated networks $\mathcal{D}$ is determined by*

$$\mathrm{Re}\lambda_2(L(\mathcal{D}))=\min\{\mathrm{Re}\lambda_2(L(\mathcal{D}_0),m_{in}(1),\dots,m_{in}(N-m_0)\},$$

where $m_{in}(t)$ is the number of in-edges of the new node added at iteration $t\in[1,N-m_0]$.

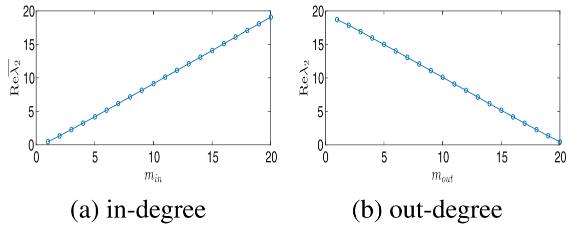


Fig. 4. The impacts of minimum (a) in-degree and (b) out-degree on algebraic connectivity Re$\overline{\lambda_2}$ (averaged over 100 simulation runs).

Proof. Similar to the proof of Theorem 2, it is clear that the Laplacian matrix $L(\mathcal{D})$ is represented as

$$L(\mathcal{D}) = \begin{pmatrix} L(\mathcal{D}_0) & O \\ * & \Delta \end{pmatrix},$$

where $\Delta = \mathrm{diag}(m_{in}(1), \ldots, m_{in}(N - m_0))$ is a diagonal matrix and its eigenvalues are $m_{in}(1), \ldots, m_{in}(N - m_0)$. Therefore, the algebraic connectivity of $\mathcal{D}$ is determined by the smallest value among $\mathrm{Re}\lambda_2(L(\mathcal{D}_0)), m_{in}(1), \ldots, m_{in}(N - m_0)$. □

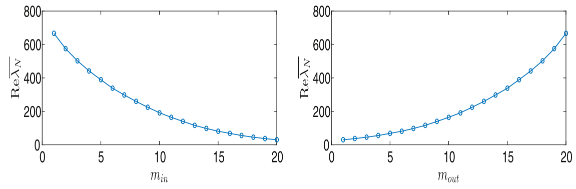
Second (exponents of in/out-degree distributions), we consider the same initial network as above, but change m_{in}, m_{out} to obtain different γ_{in}, γ_{out} (see Table 1). γ_{in}, γ_{out} reflect 'degrees' of preferential attachment, the second main feature of scale-free networks.

In Fig. 3 each plotted point is an average of 100 simulation runs. Observe that algebraic connectivity increases (resp. decreases) as the exponent of in-degree (resp. out-degree) distribution increases, consistently for different network sizes. This impact of the exponent of in-degree distribution on the algebraic connectivity is the same as that of the exponent of degree distribution in the undirected case [7]. What is interesting in the current directed networks is that the impact of the exponent of out-degree distribution is in the reverse direction. Hence for fast diffusion, it is desired to have high exponent of in-degree distribution and low exponent of out-degree distribution.

Third (minimum in/out-degree), we study the impact of minimum in/out-degree on algebraic connectivity; this is for comparison with [7] on the undirected scale-free case. For this study we set the complete graph with $m_0 = 21$ as the initial graph and increase m_{in} with the constraint $m_{in} + m_{out} = 21$.

In Fig. 4 each plotted point is an average of 100 simulation runs. Observe that algebraic connectivity increases (resp. decreases) as the minimum in-degree (resp. minimum out-degree) increases. The impact of the minimum in-degree on algebraic connectivity is the same as that of the minimum degree in the undirected case [7], while that of the minimum out-degree is in the reverse direction.

Remark 1 (Topological impacts on λ_n). In [7], the relation is studied between the minimum degree and the maximum eigenvalue λ_n, which measures the robustness against time delay in the undirected scale-free case. While it is obscure whether λ_n also determines the robustness against time delay in the directed scale-free case, we investigate the relation between the minimum in/out degree and the maximum real part of the eigenvalues, i.e. Reλ_n, for comparison with the results in [7]. As shown in Fig. 5(b), we observe that Re$\overline{\lambda_n}$ (averaged over 100 simulation runs) increases as the minimum out-degree increases, which corresponds to



(a) in-degree (b) out-degree

Fig. 5. The impacts of minimum (a) in-degree and (b) out-degree on the maximum real part of the eigenvalues $\mathrm{Re}\overline{\lambda_n}$ (averaged over 100 simulation runs).

the impact of the minimum degree in the undirected case [7]. In contrast, the impact of the minimum in-degree is in the reverse direction (see Fig. 5(a)).

4.2. Directed scale-free versus directed small-world

We compare the directed scale-free networks generated by the DSF Algorithm with another well-known model, directed small-world networks, which has been shown to have high algebraic connectivity [14].

To generate directed small-world networks, we use the following procedure (cf. [14]):

1. Let the initial network be a (regular) ring graph of N nodes, where each node has $\frac{k}{2}$ (k an even integer) nearest neighbors on both sides.
2. With probability p each edge is rewired to another node. No self-loop edges or multiple edges are allowed.

The number of edges in the generated small-world networks is constantly Nk. Recall from Eq. (3) that the number of edges in the scale-free networks generated by our DSF Algorithm is approximately $N(m_{in} + m_{out})$. Hence in the comparison simulation below, we choose $m_{in} = m_{out} = 5$ and $k = 10$, such that the number of edges of the two types of networks are roughly the same.

Moreover, for small-world networks we choose the rewiring probability $p = 0.1$ (which results in most evident small-world characteristic), and for scale-free network we choose the initial graph to be a ring graph with $m_0 = 4$ nodes.

The simulation result is displayed in Fig. 6; each plotted point is an average of 100 simulation runs. As we can see, the algebraic connectivity of directed small-world networks decreases faster as the number of nodes increases, as compared to directed scale-free networks. Further, for roughly the same number of edges, directed scale-free networks have higher algebraic connectivity than directed small-world networks. These together suggest that scale-free networks be a better model than small-world networks for scalable, faster diffusion.

4.3. Convergence speed of reaching consensus

Finally we use a concrete diffusion process, the consensus problem, to illustrate that higher algebraic connectivity leads to faster diffusion speed.

Consider a network of N agents with inter-agent communication topology $\mathcal{D} = (V, E)$. Denote the state of each agent at time t by $x_i(t)$, $i \in [1, N]$; the consensus problem is for every

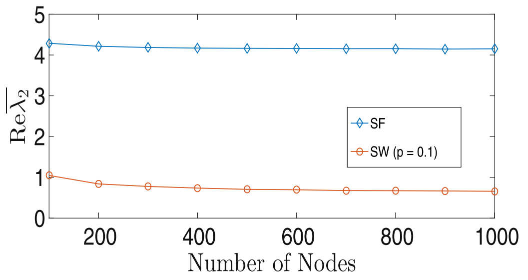


Fig. 6. Comparison between scale-free networks and small-world networks on their algebraic connectivity $\mathrm{Re}\overline{\lambda_2}$ (averaged over 100 simulation runs).

Table 2
Parameter settings for Fig. 6.

Network	Parameters
Scale-free	$m_0 = 5, m_{in} = 5, m_{out} = 5$
Small-world	$p = 0.1, k = 10$

$x_i(t)$ to asymptotically converge to a common (consensus) value, say x^*. In other words, the consensus value x^* diffuses across the network.

It is well-known that the standard consensus protocol is written as [10]

$$\dot{x}(t) = -L(\mathcal{D})x(t) \tag{11}$$

where $x(t) := [x_1(t) \cdots x_N(t)]^\top$ and $L(\mathcal{D})$ is the Laplacian matrix of the network $\mathcal{D}$. By Eq. (11), consensus is achieved if and only if the network $\mathcal{D}$ contains a spanning tree [27]. When consensus is achieved, the convergence speed to consensus is determined by $\mathrm{Re}\lambda_2(L(\mathcal{D}))$, i.e. the algebraic connectivity. In particular, the higher algebraic connectivity, the faster convergence speed.

For the same setting of Section 4-B above and choosing the same initial conditions $x(0)$ that are drawn uniformly at random from the interval $[-1, 1]$, we calculate by simulation the convergence time of Eq. (11) for directed scale-free networks and directed small-world networks. We consider the network reaches consensus if the *disagreement variable* $e(t) = x(t)^T L(\mathcal{D})x(t)$ becomes less than 1. For comparison we also include a regular graph with the number of neighbors $k = 10$ (this can be obtained from the small-world procedure by setting $p = 0$).

The simulation result is displayed in Fig. 7; each plotted point is an average of 100 simulation runs. Observe that the convergence time of directed scale-free networks and directed small-world networks are both (roughly) $O(\log N)$, which is much faster than the regular networks whose convergence time is known to be $O(N^2)$ [28]. Consistent with the algebraic connectivity comparison presented in Section 4-B above, the directed scale-free networks have faster convergence speed than directed small-world networks.

5. Conclusions

We have proposed an algorithm, extending that of Barabasi and Albert [5], to generate directed scale-free networks. Using this algorithm, we have investigated by simulations the impacts of structural properties of directed scale-free networks (size, exponents of in/out-

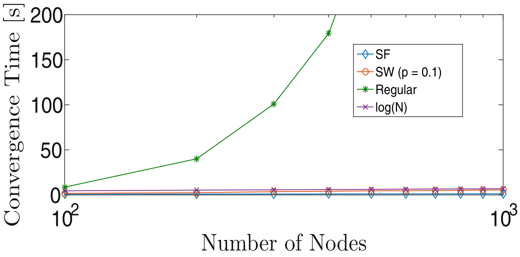


Fig. 7. Convergence speed comparison: directed scale-free, directed small-world, and regular networks. The horizontal axis (number of nodes) is in the logarithmic scale.

degree distributions, minimum in/out-degree) on the algebraic connectivity. Moreover, we have compared directed scale-free networks with directed small-world networks, and demonstrated that for the same number of nodes and comparable number of edges, directed scale-free networks have larger algebraic connectivity. Finally we have studied a representative example of diffusion processes over networks, the consensus problem, to illustrate that the speed of diffusion over directed scale-free networks is faster than that over directed small-world networks and regular networks.

In future work, we aim to investigate the algebraic connectivity of hierarchical networks [9], which have both scale-free and small-world property. In addition, we aim to derive theoretical results on algebraic connectivity of scale-free networks, following an approach to bounding algebraic connectivity by network diameter [20].

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