Top-Down Synthesis of Multiagent Formation Control: An Eigenstructure Assignment Based Approach

Takatoshi Motoyama and Kai Cai, *Senior Member, IEEE*

***Abstract*—We propose a top-down approach for formation control of heterogeneous multiagent systems, based on the method of *eigenstructure assignment*. Given the problem of achieving scalable formations on the plane, our approach globally computes a state feedback control that assigns desired closed-loop eigenvalues/eigenvectors. We characterize the relation between the eigenvalues/eigenvectors and the resulting interagent communication topology, and design special (sparse) topologies such that the synthesized control may be implemented locally by the individual agents. Moreover, we present a hierarchical synthesis procedure that significantly improves computational efficiency. Finally, we extend the proposed approach to achieve fixed-size formation and circular motion, and illustrate these results by simulation examples.**

***Index Terms*—Eigenstructure assignment, formation control, multiagent systems.**

I. Introduction

COOPERATIVE control of multiagent systems has been an active research area in the systems control community [1]–[5]. Among many problems, *formation control* has received much attention [6], [7] owing to its wide applications, such as satellite formation flying, search and rescue, terrain exploration, and foraging. A main problem studied is stabilization to a *rigid* formation, where the goal is to steer the agents to achieve a formation with a specified size and only freedoms of translation and rotation [6], [8]–[11]. Achieving a *scalable* formation with unspecified size (i.e., freedoms of scaling, translation, and rotation) has also been studied [12], [13]; a scalable formation may allow the group to adapt to unknown environment with obstacles. More recently, higher dimensional formations have been addressed via the affine formation approach [14] and the bearing-based approach [15]. In addition, methods of controlling formations in motion are presented in [16]–[18].

These different approaches/methods for formation control have a common feature in design: namely *bottom-up*. Specifically, the interagent communication topology (or a certain topological condition) is given *a priori*, which defines the neighbors for each agent. Then, based only on the neighborhood information, local control strategies are designed for the individual agents. The properties of the designed local strategies are finally analyzed at the systemic (i.e., global) level, and correctness is proved under certain graphical conditions on the communication topology. This bottom-up design is indeed the mainstream approach for cooperative control of multiagent systems that places emphasis on *distributed control*.

In this paper, we propose a distinct, *top-down* approach for formation control, based on a known method called *eigenstructure assignment* [19]–[22]. Different from the bottom-up approach, there need not be any communication topology imposed *a priori* (in fact the agents are typically assumed independent, i.e., uncoupled), and no design will be done on the local level. Indeed, given a multiagent formation control problem characterized by specific eigenvalues and eigenvectors (precisely defined in Section II), our approach constructs on the global level a feedback matrix (if it exists) that renders the closed-loop system to possess those desired eigenvalues/eigenvectors, thereby achieving desired formations. Moreover, the synthesized feedback matrix (its off-diagonal entries being zero or nonzero) defines the interagent communication topology; namely the topology is a *result* of control synthesis, which is different from the bottom-up approach where the topology (or a certain topological condition) is given *a priori*. Accordingly, the computed feedback control may be implemented by individual agents; thus our approach features "compute globally, implement locally."

Although our method requires centralized computation of control gain matrices, we show that a straightforward extension of the approach to a *hierarchical* synthesis procedure significantly reduces computation time. Empirical evidence is provided to show the efficiency of the proposed hierarchical synthesis procedure; in particular, computation of a feedback control for a group of 1000 agents needs merely a fraction of a second, which is likely to suffice for many practical purposes.

The contributions of this paper are summarized as follows.

1) Our proposed top-down approach is *systematic*, in the sense that it treats different cooperative control specifications (characterizable by desired eigenstructure) by

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The authors are with the Department of Electrical and Information Engineering, Osaka City University, Osaka 558-8585, Japan (e-mail: motoyama@c.info.eng.osaka-cu.ac.jp; kai.cai@eng.osaka-cu.ac.jp).

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the same synthesis procedure. We show that consensus, scalable formation, fixed-size formation, and cooperative circular motion can all be addressed using the same method. Moreover, several recently developed formation control methods—the complex matrix based [13], the bearing-based [15], and the affine formation [14]—can all be treated uniformly using our top-down approach.

2) We present a characterization of the relation between the resulting communication topology and the eigenstructure selected for the control synthesis. Furthermore, we show that by appropriately choosing eigenvalues and the corresponding eigenvectors, special topologies (star, cyclic, line) can be designed, and the computed feedback control may be implemented locally over these (sparse) topologies.
3) Our method is amenable to deal with not only heterogeneous agent dynamics, but even the case where some agents are not self-stabilizable (in the standard sense that these agents cannot stabilize themselves by using only their own control inputs; see the precise definition in Section IV-B).

We note that Wu and Iwasaki [23] also proposed an eigenstructure assignment method and applied it to the multiagent consensus problem. Their approach is bottom-up: first eigenstructure assignment is applied to design a *local* controller for each agent, and then these local controllers are connected according to an imposed communication topology; the correctness of this approach is established on the global level. By contrast, our approach is top-down: at the setup we do not consider an imposed topology; instead, we characterize the relation between the chosen eigenstructures and the resulting topologies, and design special topologies by selecting particular eigenstructures. In addition, we use eigenstructure assignment involving nonconjugate complex eigenvalues and eigenvectors (as opposed to real eigenvalues and eigenvectors studied in [23]), which allows us to solve scalable/fixed-size formation and circular motion on the plane.

Finally, this paper differs from its conference precursor [24] in the following aspects.

1) A precise relation between eigenstructure and topology is characterized (see Theorem 2 in Section III).
2) More general cases where the initial interagent topology is arbitrary and/or there exist nonstabilizable agents are addressed (see Section IV).
3) The problem of achieving cooperative circular motion is solved (see Section VI-B).

The rest of this paper is organized as follows. In Section II, we formulate the multiagent formation control problem. In Section III, we solve the problem by eigenstructure assignment, and discuss the relations between eigenvalues/eigenvectors and topologies. In Section IV, we study the more general cases where the initial interagent topology is arbitrary and/or there exist nonstabilizable agents. In Section V, we present a hierarchical synthesis procedure to reduce computation time, and Section VI extends the method to achieve fixed-size formation and circular motion. Simulation examples are given in Section VII and our conclusions are stated in Section VIII.

II. Problem Formulation

Consider a heterogeneous multiagent system where each agent is modeled by a first-order ordinary differential equation

$$\dot{x}_i = a_i x_i + b_i u_i, \quad i = 1, \ldots, n. \tag{1}$$

Here, $x_i \in \mathbb{C}$ is the state variable, $u_i \in \mathbb{C}$ is the control variable, and $a_i \in \mathbb{R}$ and $b_i (\neq 0) \in \mathbb{R}$ are constant parameters. Thus, each agent is a point mass moving on the complex plane, with possibly stable ($a_i < 0$), semistable ($a_i = 0$), or unstable ($a_i > 0$) dynamics. The requirement $b_i \neq 0$ is to ensure stabilizability/controllability of (a_i, b_i); thus, each agent is stabilizable/controllable. Note that represented by (1), the agents are independent (i.e., uncoupled) and *no* interagent topology is imposed at this stage.

In vector-matrix form, the system of n independent agents is given by

$$\dot{x} = Ax + Bu \tag{2}$$

where $x := [x_1 \cdots x_n]^\top \in \mathbb{C}^n$, $u := [u_1 \cdots u_n]^\top \in \mathbb{C}^n$, $A := \mathrm{diag}(a_1, \ldots, a_n)$, and $B := \mathrm{diag}(b_1, \ldots, b_n)$; here, $\mathrm{diag}(\cdot)$ denotes a diagonal matrix with the specified diagonal entries. Consider modifying (2) by a state feedback $u = Fx$ and thus the closed-loop system is given by

$$\dot{x} = (A + BF)x. \tag{3}$$

In this paper, the entries of F are generally complex, i.e., $F \in \mathbb{C}^{n\times n}$. Straightforward calculation shows that the diagonal entries of $A + BF$ are $a_i + b_i F_{ii}$, and the off-diagonal entries $b_i F_{ij}$. Since $b_i \neq 0$, the off-diagonal entries $(A + BF)_{ij} \neq 0$ if and only if $F_{ij} \neq 0\,(i \neq j)$.

In view of the structure of $A + BF$, we can define a corresponding *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows: the *node* set $\mathcal{V} := \{1, \ldots, n\}$ with node $i \in \mathcal{V}$ standing for agent i (or state x_i) and the *edge* set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with edge $(j, i) \in \mathcal{V}$ if and only if F's off-diagonal entry $F_{ij} \neq 0$. Since $F_{ij} \neq 0$ implies that agent i uses $F_{ij} x_j$ (real axis information $\mathrm{Re}(F_{ij} x_j)$ and imaginary axis information $\mathrm{Im}(F_{ij} x_j)$) in updating its state x_i, we say for this case that agent j *communicates* its state x_j to agent i, or j is a *neighbor* of i. The graph $\mathcal{G}$ is therefore called a communication network among agents, whose topology is decided by the off-diagonal entries of F. Thus, the communication topology emerges as the result of applying the state feedback control $u = Fx$.

Now, we define the formation control problem of the multiagent system (2).

Problem 1: Consider the multiagent system (2) and specify a vector $f \in \mathbb{C}^n$ ($f \neq 0$). Design a state feedback control $u = Fx$ such that for every initial condition $x(0)$, $\lim_{t\to\infty} x(t) = cf$ for some constant $c \in \mathbb{C}$.

In Problem 1, the specified vector f represents a desired *formation configuration* in the (complex) plane.[1] By formation configuration, we mean that the geometric information of the formation remains when scaling and rotational effects are

[1] We limit our attention to two-dimensional formations in this paper, and will investigate in future work three and higher dimensional formations as in [14] and [15].

discarded. Indeed, by writing the constant $c \in \mathbb{C}$ in the polar coordinate form (i.e., $c = \rho e^{j\theta}$, $j = \sqrt{-1}$), the final formation cf is the configuration f scaled by ρ and rotated by θ. The constant c is unknown *a priori* and in general depends on the initial condition $x(0)$.

Notation: Given a matrix M, ImM denotes the *image* of M over the complex field, i.e., the complex span of the column vectors of M. Similarly, Ker$M$ denotes the *kernal* of M over the complex field.

III. Main Results

In this section we solve Problem 1, the formation control problem, based on the method of *eigenstructure assignment* [19]. The following is our main result.

Theorem 1: Consider the multiagent system (2) and let f be a desired formation configuration. Then, there always exists a state feedback control $u = Fx$ that solves Problem 1, i.e.,

$$(\forall x(0) \in \mathbb{C}^n)(\exists c \in \mathbb{C}) \lim_{t\to\infty} x(t) = cf.$$

Proof: By standard linear systems theory, the multiagent system (2) with $u = Fx$ achieves a formation configuration $f \in \mathbb{C}^n$ if the closed-loop matrix $A + BF$ has the following eigenstructure.

1) Its eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfy

$$0 = \lambda_1 < |\lambda_2| \leq \cdots \leq |\lambda_n|, \text{ and } (\forall i \in [2, n])\text{Re}(\lambda_i) < 0. \tag{4}$$

2) The corresponding eigenvectors

$$\{v_1, v_2, \ldots, v_n\} \text{ are linearly independent, and } v_1 = f. \tag{5}$$

So we must verify that the above eigenstructure is assignable by state feedback $u = Fx$ for the multiagent system (2). Note that except for (λ_1, v_1), which is fixed, we have freedom to choose (λ_i, v_i), $i \in [2, n]$. Thus, for simplicity, we let λ_i be all distinct.

In (2), we have $A = \text{diag}(a_1, \ldots, a_n)$, $B = \text{diag}(b_1, \ldots, b_n)$, and $b_i \neq 0$ for all $i \in [1, n]$. Thus, it is easily checked that the pair (A, B) is controllable and Ker$B = 0$. To show that there exists F such that $(A + BF)v_i = \lambda_i v_i$, for each $i \in [1, n]$ (with λ_i, v_i specified previously), it is necessary and sufficient [19] to verify the condition $v_i \in \text{Im}N_1(\lambda_i)$, where $N_1(\lambda_i)$ satisfies

$$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} N_1(\lambda_i) \\ N_2(\lambda_i) \end{bmatrix} = 0. \tag{6}$$

First, for $\lambda_1 = 0$, we find a basis for

$$\text{Ker}[\lambda_1 I - A \;\; B] = \text{Ker}[-A \;\; B]$$

and derive $N_1(\lambda_1) = B$ and $N_2(\lambda_1) = A$. So Im$N_1(\lambda_1) = \mathbb{C}^n$, and hence, $v_1 = f \in \text{Im}N_1(\lambda_1)$.

Next, let $i \in [2, n]$; we find a basis for Ker$[\lambda_i I - A \;\; B]$ and derive $N_1(\lambda_i) = B$ and $N_2(\lambda_i) = A - \lambda_i I$. Thus, again Im$N_1(\lambda_i) = \mathbb{C}^n$, and $v_i \in \text{Im}N_1(\lambda_i)$. Therefore, we conclude that there always exists a state feedback $u = Fx$ such that the multiagent system (2) achieves the formation configuration f. ■

In the proof, we showed that the desired eigenstructure (4), (5) for solving the formation control Problem 1 is assignable to the closed-loop matrix $A + BF$, whose null space (the eigenspace corresponding to the eigenvalue 0) is consequently spanned by $f \in \mathbb{C}^n$ (over the complex field). Different null spaces of $A + BF$ in fact correspond to many cooperative control problems and several new approaches to formation control, which are as follows.

1) For the consensus problem, the null space is span$\{\mathbf{1}\}$ (where $\mathbf{1} := [1 \cdots 1]^\top \in \mathbb{R}^n$).
2) For flocking of double-integrator agents, it is span$\{[\mathbf{1}^\top \; 0]^\top\}$.
3) For the complex matrix based approach to formation control [13], it is span$\{\mathbf{1}, f\}$.
4) For the bearing-based approach [15], it is span$\{\mathbf{1} \otimes I_d, f_{\text{real}}\}$ (with $d > 1$ and $f_{\text{real}} \in \mathbb{R}^{dn}$).
5) For the affine formation [14], it is the space of all the affine images of $f_{\text{real}} \in \mathbb{R}^{dn}$.

As a result, these different cooperative control problems and approaches may be treated uniformly using our eigenstructure assignment based method.

Theorem 1 asserts that the formation control Problem 1 always has a solution $u = Fx$. Moreover, by [19], the feedback matrix F may be computed by the following formula:

$$F = [w_1 \cdots w_n][v_1 \cdots v_n]^{-1} \tag{7}$$

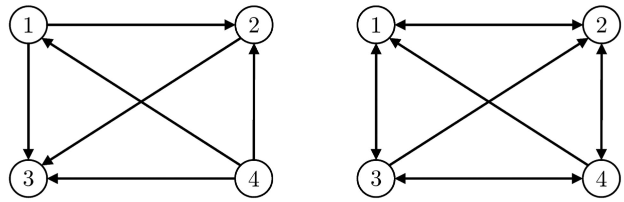
where $w_i = -N_2(\lambda_i)k_i$, $N_1(\lambda_i)k_i = v_i$, and $N_1(\lambda_i), N_2(\lambda_i)$ in (6), for all $i \in [1, n]$. This computation of F has complexity $O(n^3)$, inasmuch as the calculations involved are solving systems of linear equations, matrix inverse, and multiplication (e.g., [25]). Thus, the computational cost becomes expensive as the number of agents increases. To address this issue, we provide in Section V below a more efficient *hierarchical* procedure for synthesizing F.

The computed feedback matrix F in turn gives rise to the agents' communication graph $\mathcal{G}$. The following is an illustrative example.

Example 1: Consider the multiagent system (2) of four single integrators (that is, $a_i = 0$ and $b_i = 1$, $i = 1, \ldots, 4$).

i) *Square formation* with $f = [1 \;\; j \;\; -1 \;\; -j]^\top$ ($j = \sqrt{-1}$). Let the desired closed-loop eigenvalues be $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$, and $\lambda_4 = -3$ and the corresponding eigenvectors be $v_1 = f$, $v_2 = [1 \;\; 1 \;\; 0 \;\; 0]^\top$, $v_3 = [0 \;\; 1 \;\; 1 \;\; 0]^\top$, and $v_4 = [0 \;\; 0 \;\; 1 \;\; 0]^\top$. By (7), one computes the control gain matrix F_1, which determines the corresponding communication graph $\mathcal{G}_1$ (see Fig. 1). Observe that F_1 contains complex entries, which may be viewed as control gains for the real and imaginary axes, respectively, or scaling and rotating gains on the complex plane. Also note that $\mathcal{G}_1$ has a spanning tree with node 4 the root, and the computed feedback control $u = Fx$ can be implemented by the four agents individually.

ii) *Consensus* with $f = [1 \; 1 \; 1 \; 1]^\top$. Let the desired eigenvalues be $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$, and $\lambda_4 = -4$ and the corresponding eigenvectors be $v_1 = f$, $v_2 = [1 \;\; 1 \;\; 0 \;\; 1]^\top$, $v_3 = [1 \;\; 0 \;\; 0 \;\; 1]^\top$, and $v_4 = [0 \;\; 0 \;\; 1 \;\; -1]^\top$. Again



$$F_1 = \begin{bmatrix} -1 & 0 & 0 & j \\ 1 & -2 & 0 & -2-j \\ -1 & 1 & -3 & 1-2j \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} -4 & 2 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -4 & 0 & 0 & 4 \\ 0 & 2 & 1 & -3 \end{bmatrix}$$

Fig. 1. Example: feedback matrices and communication topologies.

by (7), one computes the control gain matrix F_2 and the corresponding graph $\mathcal{G}_2$ (see Fig. 1).

Note that in this case, F_2 is real and $\mathcal{G}_2$ strongly connected. But unlike the usual consensus algorithm (e.g., [2]), $-F_2$ is not a *graph Laplacian matrix* for the entries $(2,1)$ and $(3,1)$ are positive. Thus, our eigenstructure assignment based approach may generate a larger class of consensus algorithms with negative weights.

We remark that in our approach, the *convergence speed* to the desired formation configuration is assignable. This is because the convergence speed is dominated by the eigenvalue λ_2, with the second largest real part, of the closed-loop system $\dot{x} = (A + BF)x$; and in our approach λ_2 is freely assignable. The smaller the $\mathrm{Re}(\lambda_2)$ is, the faster the convergence to formation occurs (at the cost of higher control gain). As an example, for (ii) in Example 1 assign the second largest eigenvalue $\lambda_2 = -2$ (originally -1), and change $v_4 = [0\ 0\ \frac{1}{2}\ \ -1]$. This results in a new feedback matrix

$$F_2' = \begin{bmatrix} -4 & 1 & 2 & 1 \\ -1 & -2 & 2 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

which has zero entries at the same locations as F_2. Thus, with the same topology, F_2' achieves faster convergence speed.

As we have seen in Example 1, the feedback matrix F's off-diagonal entries, which determine the topology of $\mathcal{G}$, are dependent on the choice of eigenvalues as well as eigenvectors. Namely, different sets of eigenvalues and eigenvectors result in different interagent communication topologies. Our next result characterizes a precise relation between the eigenvalues/eigenvectors and the topologies.

Theorem 2: Consider the multiagent system (2) and f a desired formation configuration. Let the eigenstructure λ_i and v_i $(i = 1,\ldots,n)$ be as in (4) and (5), and denote the rows of $[v_1 \cdots v_n]^{-1}$ by v_i^*. Then, the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the (closed-loop) multiagent system is such that

$$(i_1, j_1), \ldots, (i_K, j_K) \notin \mathcal{E} \quad (K \geq 1)$$

if and only if the vector

$$[\lambda_2 \ \cdots \ \lambda_n]^\top$$

is orthogonal to the subspace spanned by the following K vectors:

$$\left[v_{2\,i_1} v^*_{2\,j_1} \ \cdots \ v_{n\,i_1} v^*_{n\,j_1}\right]^\top, \ldots, \left[v_{2\,i_K} v^*_{2\,j_K} \ \cdots \ v_{n\,i_K} v^*_{n\,j_K}\right]^\top.$$

Proof: For each λ_i, $i \in [1,n]$, we derive from (6) that

$$(\lambda_i I - A) N_1(\lambda_i) + B N_2(\lambda_i) = 0.$$

Choose $N_1(\lambda_i) = B$ and $N_2(\lambda_i) = -(\lambda_i I - A)$ to satisfy the above equation. Then, $k_i = N_1^{-1}(\lambda_i) v_i = B^{-1} v_i$ and $w_i = -N_2(\lambda_i) k_i = (\lambda_i I - A) B^{-1} v_i$. By (7), we have

$$\begin{aligned} F &= [w_1 \cdots w_n][v_1 \cdots v_n]^{-1} \\ &= [(\lambda_1 I - A) B^{-1} v_1 \cdots (\lambda_n I - A) B^{-1} v_n][v_1 \cdots v_n]^{-1} \\ &= B^{-1}(-A[v_1 \cdots v_n] + [\lambda_1 v_1 \cdots \lambda_n v_n])[v_1 \cdots v_n]^{-1} \\ &= -B^{-1} A + B^{-1}[v_1 \cdots v_n] \mathrm{diag}(\lambda_1, \ldots, \lambda_n)[v_1^* \cdots v_n^*]^\top. \end{aligned}$$

Thus, the closed-loop matrix is given by

$$A + BF = [v_1 \cdots v_n] \mathrm{diag}(\lambda_1, \ldots, \lambda_n)[v_1^* \cdots v_n^*]^\top. \tag{8}$$

The (i,j)-entry of $A + BF$ is

$$\begin{aligned} (A+BF)_{ij} &= \lambda_1 v_{1i} v_{1j}^* + \lambda_2 v_{2i} v_{2j}^* + \cdots + \lambda_n v_{ni} v_{nj}^* \\ &= \lambda_2 v_{2i} v_{2j}^* + \cdots + \lambda_n v_{ni} v_{nj}^* \quad (\lambda_1 = 0) \\ &= [\lambda_2 \ \cdots \ \lambda_n][v_{2i} v_{2j}^* \ \cdots \ v_{ni} v_{nj}^*]^\top. \end{aligned}$$

Therefore, $(A+BF)_{i_1 j_1} = \cdots = (A+BF)_{i_K j_K} = 0$, i.e., in the communication graph $(i_1, j_1), \ldots, (i_K, j_K) \notin \mathcal{E}$, if and only if the vector $[\lambda_2 \ \cdots \ \lambda_n]^\top$ is orthogonal to each of the following K vectors:

$$\left[v_{2\,i_1} v^*_{2\,j_1} \ \cdots \ v_{n\,i_1} v^*_{n\,j_1}\right]^\top, \ldots, \left[v_{2\,i_K} v^*_{2\,j_K} \ \cdots \ v_{n\,i_K} v^*_{n\,j_K}\right]^\top.$$

Namely, $[\lambda_2 \ \cdots \ \lambda_n]^\top$ is orthogonal to the subspace spanned by these K vectors. ■

Once the desired eigenvalues and eigenvectors are chosen, Theorem 2 provides a necessary and sufficient condition to check the interconnection topology among the agents, without actually computing the feedback matrix F. On the other hand, the problem of choosing an appropriate eigenstructure to match a given topology is more difficult, inasmuch as there are many free variables to be determined in the eigenvalues and eigenvectors. While we shall investigate the general problem of eigenstructure design for imposing particular topologies in our future work, in the following section, nevertheless, we show that choosing certain appropriate eigenstructures results in certain special (sparse) topologies. With these topologies, the synthesized control $u = Fx$ may be implemented in a distributed fashion.

A. Special Topologies

We show how to derive three special types of topologies (see Fig. 2) by choosing appropriate eigenstructures. Due to space limit the proofs of this section are referred to [26].

1) Star Topology: A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a star topology if there is a single root node, say node 1, and $\mathcal{E} = \{(1,i) | i \in [2,n]\}$. Thus, all the other nodes receive information

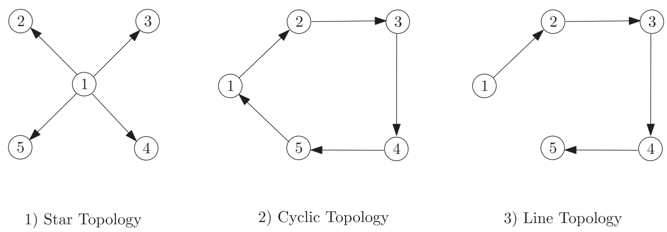


Fig. 2. Special topologies (five node examples). (a) Star topology. (b) Cyclic topology. (c) Line topology.

from, and only from, the root node 1. In terms of the total number of edges, a star topology is one of the sparsest topologies, with the least number $(n-1)$ of edges, that contain a spanning tree. Now, consider the following eigenstructure:

$$\begin{aligned}&\text{eigenvalues: } \lambda_1 = 0, \lambda_2, \ldots, \lambda_n \text{ distinct}\\&\qquad\qquad \text{and } \mathrm{Re}(\lambda_2), \ldots, \mathrm{Re}(\lambda_n) < 0\\&\underset{\text{(independent)}}{\text{eigenvectors: } [v_1\ v_2 \cdots v_n]} = \begin{bmatrix} f_1 & 0 & \cdots & 0\\ f_2 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ f_n & 0 & \cdots & 1\end{bmatrix}. \end{aligned} \tag{9}$$

Proposition 1: Consider the multiagent system (2). If the eigenstructure (9) is used in the synthesis of feedback control $u = Fx$, then Problem 1 is solved and the resulting graph $\mathcal{G}$ is a star topology.

2) Cyclic Topology: A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a cyclic topology if $\mathcal{E} = \{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\}$. Consider the following eigenstructure:

$$\begin{aligned}&\text{eigenvalues: } \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{0, \omega - 1, \ldots, \omega^{n-1} - 1\}\\&\text{eigenvectors: } [v_1\ v_2 \cdots v_n] \ \text{(independent)}\\&= \begin{bmatrix} f_1 & f_1 & \cdots & f_1\\ f_2 & f_2\omega & \cdots & f_2\omega^{n-1}\\ f_3 & f_3\omega^2 & \cdots & f_3\omega^{2(n-1)}\\ \vdots & \vdots & \vdots & \vdots\\ f_n & f_n\omega^{n-1} & \cdots & f_n\omega^{(n-1)(n-1)}\end{bmatrix}\end{aligned} \tag{10}$$

where $\omega := e^{2\pi j/n}$ $(j = \sqrt{-1})$.

Proposition 2: Consider the multiagent system (2). If the eigenstructure (10) is used in the synthesis of feedback control $u = Fx$, then Problem 1 is solved and the resulting $\mathcal{G}$ is a cyclic topology.

3) Line Topology: A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a (directed) line topology if there is a single root node, say node 1, and $\mathcal{E} = \{(1,2), (2,3), \ldots, (n-1, n)\}$. A line topology is also one of the sparsest topologies containing a spanning tree. Now, consider the following eigenstructure:

$$\begin{aligned}&\text{eigenvalues: } \lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = -1\\&\underset{\text{(independent)}}{\text{eigenvectors: } [v_1\ v_2 \cdots v_n]} = \begin{bmatrix} f_1 & 0 & \cdots & 0\\ f_2 & 0 & \cdots & -f_2\\ \vdots & \vdots & & \vdots\\ f_{n-1} & 0 & \ddots & -f_{n-1}\\ f_n & -f_n & \cdots & -f_n\end{bmatrix}. \end{aligned} \tag{11}$$

Proposition 3: Consider the multiagent system (2). If the eigenstructure (11) is used in the synthesis of feedback control $u = Fx$, then Problem 1 is solved and the resulting $\mathcal{G}$ is a line topology.

Remark: While the eigenstructures in Propositions 1–3 are sufficient to ensure the respective graph topologies, they need not to be necessary in general. For example, let $n = 3$, $a_1 = a_2 = a_3 = 0$, $b_1 = b_2 = b_3 = 1$, and $f = [1\ 2\ 3]^\top$. Consider a state feedback control $u = Fx$, where

$$F = \begin{bmatrix} 0 & 0 & 0\\ 2 & -1 & 0\\ 3 & 0 & -1\end{bmatrix} \left(\text{resp. } F = \begin{bmatrix} -2 & 1 & 0\\ 0 & -3 & 2\\ 3 & 0 & -1\end{bmatrix}\right.$$
$$\left.\text{or } F = \begin{bmatrix} 0 & 0 & 0\\ 2 & -1 & 0\\ 0 & 3 & -2\end{bmatrix}\right).$$

It is easily verified that Problem 1 is solved by $u = Fx$ and the closed-loop matrix $A + BF$ corresponds to a star (resp. cyclic or line) topology. However, the eigenstructure of $A + BF$ is different from (9) [resp. (10) or (11)] .

IV. General Multiagent Systems

So far, we have considered the multiagent system in (2), where the agents are uncoupled and each is (self) stabilizable (the matrices A and B are diagonal and B's diagonal entries nonzero). For (2), we have shown in Theorem 1 that a state feedback control, based on eigenstructure assignment, always exists to drive the agents to a desired formation.

More generally, however, the agents may be initially interconnected (owing to physical coupling or existence of communication channels), and/or some agents might not be capable of stabilizing themselves (though they can receive information from others). It is thus of interest to inquire, based on the eigenstructure assignment approach, what conclusions we can draw for formation control in these more general cases.

A. Arbitrary Interagent Connections

First, we consider the case where the agents have arbitrary initial interconnection while keeping the assumption that they are individually stabilizable. That is, we consider the following multiagent system:

$$\dot{x} = Ax + Bu \tag{12}$$

where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^n$, $A \in \mathbb{R}^{n\times n}$, and $B = \text{diag}(b_1, \ldots, b_n)$ ($b_i \neq 0$). The matrix A is now an arbitrary real matrix, modeling an arbitrary (initial) communication topology among the agents.

It turns out, despite the general A matrix, that the same conclusion as Theorem 1 holds.

Theorem 3: Consider the multiagent system (12) and let f be a desired formation configuration. Then, there always exists a state feedback control $u = Fx$ that achieves formation control, i.e.,

$$(\forall x(0) \in \mathbb{C}^n)(\exists c \in \mathbb{C}) \lim_{t\to\infty} x(t) = cf.$$

The proof is similar to that of Theorem 1, with the following difference. In the proof of Theorem 1, we derived $N_1(\lambda_i) = B$ and $N_2(\lambda_i) = A - \lambda_i I$. Since A was diagonal and diagonal matrices commute, there held

$$\begin{bmatrix}\lambda_i I - A \ B\end{bmatrix}\begin{bmatrix} B \\ A - \lambda_i I\end{bmatrix} = 0.$$

For a general A as in Theorem 3, we have found instead $N_1(\lambda_i) = B$ and $N_2(\lambda_i) = B^{-1}(A - \lambda_i I)B$ that deal with arbitrary A without depending on the commutativity of matrices.

Theorem 3 asserts that, as long as the agents are individually stabilizable, formation control is achievable by eigenstructure assignment regardless of how the agents are *initially* interconnected. The *final* topology, on the other hand, is in general determined by the initial connections "plus" additional ones resulted from the chosen eigenvalues/eigenvectors (as has been discussed in Section III). It may also be possible, however, that the initial connections are "decoupled" by the corresponding entries of the synthesized feedback matrix. This is illustrated by the following example.

Consider again Example 1(i), but change A from the zero matrix to the following:

$$A = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

that is, agents 1 and 2, 3 and 1, 4 and 3 are initially interconnected. Assigning the same eigenstructure as in Example 1(i), we obtain the following feedback matrix:

$$F = \begin{bmatrix} -1 & -0.5 & 0 & j \\ 1 & -2 & 0 & -2-j \\ -0.5 & 1 & -3 & 1-2j \\ 0 & 0 & -2 & 0 \end{bmatrix}.$$

Then, the closed-loop matrix $A + BF$ (where B is the identity matrix) is

$$A + BF = \begin{bmatrix} -1 & 0 & 0 & j \\ 1 & -2 & 0 & -2-j \\ -1 & 1 & -3 & 1-2j \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the same as the feedback matrix F_1 (as well as the closed-loop matrix) in Example 1(i). Thus, despite the initial coupling, the final topology turns out to be the same as that of Example 1(i). In particular, in the final topology agents 1 and 2, 4 and 3 are *uncoupled*—their initial couplings are "canceled" by the corresponding entries of the feedback matrix F.

B. Existence of Nonstabilizable Agents

Continuing to consider arbitrary initial topology (i.e., general A), we further assume that some agents cannot stabilize themselves (i.e., the corresponding diagonal entries of B in (12) are zero). Equivalently, the nonstabilizable agents have *no* control inputs. In this case, achieving a desired formation is possible only if those nonstabilizable agents may take advantage of information received from others (via connections specified by A). This is a problem of global formation stabilization with locally unstabilizable agents, which has rarely been studied in the literature. We aim to provide an answer using our top-down eigenstructure assignment based approach.

Without loss of generality, assume that only the first m $(< n)$ agents are stabilizable. Thus, the multiagent system we consider in this section is given by

$$\dot{x} = Ax + Bu \tag{13}$$

where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$, $A \in \mathbb{R}^{n\times n}$, and

$$B = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_m \\ & & \\ & 0 & \\ & & \end{bmatrix} \in \mathbb{R}^{n\times m} \ (m < n)$$

$$b_i \neq 0, \ i \in [1, m].$$

Theorem 4: Consider the multiagent system (13) and let f be a desired formation configuration. Also, let $\lambda_1, \ldots, \lambda_n$ and $v_1, \ldots, v_n$ be the desired eigenvalues and eigenvectors satisfying (4) and (5). If the following statements holds:

i) the pair (A, B) is controllable;
ii) $\text{Im}B \subseteq \text{Im}(A - \lambda_i I)$ for all $i \in [1, n]$; and
iii) $v_i \in \text{Im}N_1(\lambda_i)$ for all $i \in [1, n]$, where $N_1(\lambda_i)$ satisfies $(A - \lambda_i I)N_1(\lambda_i) = B$

then there exists a state feedback control $u = Fx$ that achieves formation control, i.e.,

$$(\forall x(0) \in \mathbb{C}^n)(\exists c \in \mathbb{C}) \lim_{t\to\infty} x(t) = cf.$$

Proof: First observe from (13) that $\text{Ker}B = 0$, since B's columns are linearly independent. Now, let $i \in [1, n]$. Since (A, B) is controllable [condition (i)], there exist $N_1(\lambda_i)$ and $N_2(\lambda_i)$ such that (6) holds. Setting $N_2(\lambda_i) = I$, we derive from (6) the following matrix equation:

$$(A - \lambda_i I)\, N_1(\lambda_i) = B. \tag{14}$$

Since $\text{Im}B \subseteq \text{Im}(A - \lambda_i I)$ [condition (ii)], this equation has a solution $N_1(\lambda_i)$ (which is determined by A, B, λ_i). Finally, since $v_i \in \text{Im}N_1(\lambda_i)$ [condition (iii)], the condition of eigenstructure assignment [19] is satisfied. Therefore, the desired eigenvalues and eigenvectors satisfying (4) and (5) may be assigned by a state feedback control $u = Fx$, i.e., formation control is achieved. ■

Theorem 4 provides sufficient conditions to ensure solvability of the formation control problem for multiagent systems with nonstabilizable agents. In the following, we illustrate this result by working out a concrete example, where A represents a directed line topology and there is only one agent that is stabilizable (i.e., B is simply a vector).

Example 2: Consider the multiagent system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} a_1 & 0 & & 0 \\ \hat{a}_2 & a_2 & & \\ & \ddots & \ddots & \\ 0 & & \hat{a}_n & a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $a_1, \ldots, a_n, \hat{a}_2, \ldots, \hat{a}_n$, and b_1 are nonzero. Namely, A represents a directed line topology with agent 1 the root, and B means that only agent 1 is stabilizable. Thus, this is a *single-input* multiagent system—by controlling only the root of a directed line.

First, it is verified that (A, B) is controllable, i.e., condition (i) of Theorem 4 is satisfied. To ensure condition (ii), $\mathrm{Im}B \subseteq \mathrm{Im}(A - \lambda_i I)$, it suffices to choose each desired eigenvalue λ_i $(i \in [1, n])$ such that $\lambda_i \neq a_j$ for $j \in [1, n]$ (i.e., every eigenvalue is distinct from the nonzero diagonal entries of A). At the same time, these eigenvalues must satisfy (4).

Having condition (ii) hold, the following equation has a solution $N_1(\lambda_i)$:

$$\begin{bmatrix} a_1 - \lambda_i & 0 & & 0 \\ \hat{a}_2 & a_2 - \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & \hat{a}_n & a_n - \lambda_i \end{bmatrix} N_1(\lambda_i) = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solving this equation, we obtain

$$N_1(\lambda_i) = \begin{bmatrix} \frac{b_1}{a_1 - \lambda_i} \\ -\frac{\hat{a}_2 b_1}{(a_1 - \lambda_i)(a_2 - \lambda_i)} \\ \vdots \\ (-1)^{n-1} \frac{\hat{a}_2 \cdots \hat{a}_n b_1}{(a_1 - \lambda_i)(a_2 - \lambda_i) \cdots (a_n - \lambda_i)} \end{bmatrix}.$$

Hence, to ensure condition (iii) of Theorem 4, we must choose each desired eigenvector v_i $(i \in [1, n])$ such that $v_i \in \mathrm{Im}N_1(\lambda_i)$ and (5) is satisfied. In particular, for $i = 1$, we have $\lambda_i = 0$ and $v_1 = f$; thus, $v_1 \in \mathrm{Im}N_1(\lambda_1)$ means that the formation vector f must be such that

$$f = c \left[\frac{b_1}{a_1} \; - \frac{\hat{a}_2 b_1}{a_1 a_2} \cdots (-1)^{n-1} \frac{\hat{a}_2 \cdots \hat{a}_n b_1}{a_1 a_2 \cdots a_n} \right]^\top \tag{15}$$

where $c \in \mathbb{C}$ $(c \neq 0)$. This characterizes the set of all achievable formation configurations for the single-input multiagent system under consideration.

We conclude that, by controlling only one agent, indeed the root agent of a directed line topology, it is not possible to achieve arbitrary formation configurations but those determined by the nonzeros entries of the matrices A and B in the specific manner, as given in (15).

V. Hierarchical Eigenstructure Assignment

In the previous sections, we have shown that a control gain matrix F can always be computed (as long as every agent is stabilizable) such that the multiagent formation Problem 1 is solved. Computing such F by (7) has complexity $O(n^3)$, where n is the number of agents. Consequently, the computation cost becomes expensive as the number of agents increases.

To address this issue of centralized computation, we propose in this section a *hierarchical synthesis procedure*. We shall show that the control gain matrix F computed by this hierarchical procedure again solves Problem 1, which moreover significantly improves computational efficiency (empirical evidence provided in Section VII).

For clarity of presentation, let us return to consider the multiagent system (2), and Problem 1 with the desired formation configuration $f \in \mathbb{C}^n$ $(f \neq 0)$. Partition the agents into l (≥ 1) pairwise disjoint groups. Let group k $(\in [1, l])$ have n_k (≥ 1) agents; n_k may be different and $\Sigma_{k=1}^{l} n_k = n$.

Now, for the configuration f and x, u, A, and B in (2), write in accordance with the partition (possibly with reordering)

$$f = \begin{bmatrix} g_1 \\ \vdots \\ g_l \end{bmatrix}, x = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix}, u = \begin{bmatrix} w_1 \\ \vdots \\ w_l \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_l \end{bmatrix}, B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_l \end{bmatrix}$$

where $g_k, y_k, w_k \in \mathbb{C}^{n_k}$ and $A_k, B_k \in \mathbb{C}^{n_k \times n_k}$, and $k \in [1, l]$. Thus, for each group k, the dynamics is given by

$$\dot{y}_k = A_k y_k + B_k w_k. \tag{16}$$

For later use, also write g_{k1}, y_{k1}, w_{k1} (resp. A_{k1}, B_{k1}) for the first component of g_k, y_k, w_k (resp. (1,1)-entry of A_k, B_k), and $g_0 := [g_{11} \cdots g_{l1}]^\top$, $y_0 := [y_{11} \cdots y_{l1}]^\top$, $w_0 := [w_{11} \cdots w_{l1}]^\top$, $A_0 := \mathrm{diag}(A_{11}, \ldots, A_{l1})$, and $B_0 := \mathrm{diag}(B_{11}, \ldots, B_{l1})$.

The vector g_k $(k \in [1, l])$ is the *local* formation configuration for group k, whereas g_0 is the formation configuration for the set of the first component agent from each group. We assume that these configurations are all nonzero, i.e., $g_k \neq 0$ for $k \in [1, l]$ and $g_0 \neq 0$. Now, we present the hierarchical synthesis procedure.

i) For each group $k \in [1, l]$ and its dynamics (16), compute F_k by (7) such that $A_k + B_k F_k$ has a simple eigenvalue 0 with the corresponding eigenvector g_k, and other eigenvalues have negative real parts; moreover, the topology defined by F_k has a unique root node y_{k1} (e.g., star or line by the method given in Section III-A).

ii) Treat $\{y_{k1} | k \in [1, l]\}$ (the group leaders) as a higher level group, with the following dynamics:

$$\dot{y}_0 = A_0 y_0 + B_0 w_0. \tag{17}$$

Compute $F_0 \in \mathbb{C}^{l \times l}$ by (7) such that $A_0 + B_0 F_0$ has a simple eigenvalue 0 with the corresponding eigenvector g_0, and other eigenvalues have negative real parts.

iii) Set the control gain matrix $F := F^{\text{low}} + F^{\text{high}}$, where

$$F^{\text{low}} := \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_l \end{bmatrix}$$

and F^{high} is partitioned according to F^{low}, with each block (i,j), $i,j \in [1,l]$

$$(F^{\text{high}})_{ij} = (F_0)_{ij} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} (F_0)_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The computational complexity of Step i) is $O(\hat{n}^3)$, where $\hat{n} := \max\{n_1, \ldots, n_l\}$; and Step ii) is $O(l^3)$. Let $\tilde{n} := \max\{\hat{n}, l\}$. Then, the complexity of the entire hierarchical synthesis procedure is $O(\tilde{n}^3)$. With proper group partition, this hierarchical procedure can significantly reduce computation time, as demonstrated by an empirical study in Section VII.

Note that in Step i) of the previous procedure, requiring the topology defined by each F_k to have a unique root, i.e., a single leader, is for simplicity of presentation. It can be extended to the case of multiple leaders, and then in Step ii) treat all of the leaders at the higher level. On the other hand, the number of leaders should be kept small such that the high-level control synthesis in Step ii) can be done efficiently.

The graph topology resulted from the hierarchical synthesis procedure may be viewed as a hierarchical one. On the low level, the topologies of individual groups are defined by F_k ($k \in [1,l]$); on the high level, the topology of group leaders is defined by F_0.

The correctness of the hierarchical synthesis procedure is asserted in the following.

Theorem 5: Consider the multiagent system (2) and let f be a desired formation configuration. Then, the state feedback control $u = Fx$ synthesized by the hierarchical synthesis procedure solves Problem 1, i.e.,

$$(\forall x(0) \in \mathbb{C}^n)(\exists c \in \mathbb{C}) \lim_{t\to\infty} x(t) = cf.$$

Proof: For each $k \in [1,l]$, let $y'_k := [y_{k2} \ \cdots \ y_{kn_k}]^\top$ and $g'_k := [g_{k2} \ \cdots \ g_{kn_k}]^\top \in \mathbb{C}^{n_k - 1}$. Thus, y'_k and g'_k are y_k and g_k with the first element removed. By Step i) of the hierarchical synthesis procedure, since y_{k1} is the unique root node, we can write $\dot{y}_k = (A_k + B_k F_k) y_k$ as follows:

$$\begin{bmatrix} \dot{y}_{k1} \\ \hline \dot{y}'_k \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline H_k & G_k \end{array}\right] \begin{bmatrix} y_{k1} \\ \hline y'_k \end{bmatrix}.$$

Then, by the eigenstructure of $A_k + B_k F_k$, all the eigenvalues of G_k have negative real parts and

$$\left[H_k \,\middle|\, G_k \right] \begin{bmatrix} g_{k1} \\ \hline g'_k \end{bmatrix} = 0. \tag{18}$$

Reorder $x = [y_1^\top \cdots y_l^\top]^\top$ to get $\hat{x} := [y_0^\top \, y_1'^\top \cdots y_n'^\top]^\top$. Then, there is a permutation matrix that similarly transforms the control gain matrix F in Step iii) to $\hat{F}$, and $\dot{\hat{x}} = \hat{F}\hat{x}$ is

$$\begin{bmatrix} \dot{y}_0 \\ \dot{y}'_1 \\ \vdots \\ \dot{y}'_l \end{bmatrix} = \left[\begin{array}{c|ccc} A_0 + B_0 F_0 & & 0 & \\ \hline H_1 & G_1 & & \\ \ddots & & \ddots & \\ H_l & & & G_l \end{array}\right] \begin{bmatrix} y_0 \\ y'_1 \\ \vdots \\ y'_l \end{bmatrix}.$$

It then follows from the eigenstructure of $A_0 + B_0 F_0$ assigned in Step ii) and (18) that the matrix $\hat{F}$ has a simple eigenvalue 0 with the corresponding eigenvector $\hat{f} := [g_0^\top \ g_1'^\top \ \cdots \ g_l'^\top]^\top$, and other eigenvalues have negative real parts. Hence

$$(\forall \hat{x}(0) \in \mathbb{C}^n)(\exists \hat{c} \in \mathbb{C}) \lim_{t\to\infty} \hat{x}(t) = \hat{c}\hat{f}.$$

Since $\hat{x}$ (resp. $\hat{f}$) is just a reordering of x (resp. f), the conclusion follows and the proof is complete. ■

VI. Fixed-Size Formation and Circular Motion

In this section, we show that our method of eigenstructure assignment may be easily extended to address problems of fixed-size formation and circular motion.

A. Fixed-Size Formation

First, we extend our method to study the problem of achieving a formation that has translational and rotational freedom but fixed size.

Problem 2: Consider the multiagent system (2) and specify $f \in \mathbb{C}^n$ ($f \neq 0$) and $d > 0$. Design a control u such that $\lim_{t\to\infty} x(t) = c\mathbf{1} + dfe^{j\theta}$ for some $c \in \mathbb{C}$ and $\theta \in [0, 2\pi)$.

In Problem 2, the goal of the multiagent system (2) is to achieve a formation df, with translational freedom in c, rotational freedom in θ, and fixed size d. These are three important parameters for the formation to complete certain tasks and dynamically respond to the environment.

We now present the *fixed-size formation synthesis procedure*.

1) Compute F by (7) such that $A + BF$ has *two* eigenvalues 0 with the corresponding (nongeneralized) eigenvectors $\mathbf{1}$ and f, and other eigenvalues have negative real parts;[2] moreover, the topology defined by F is 2-rooted[3] with exactly 2 roots (say nodes 1 and 2). This topology may be

[2]For repeated eigenvalues with nongeneralized eigenvectors, the eigenstructure assignment result in [19], and the computation of control gain matrix F in (7) remain the same as for the case of distinct eigenvalues.

[3]A 2-rooted topology is one where there exist 2 nodes from which every other node v can be reached by a directed path after removing an arbitrary node other than v [13].

achieved by assigning appropriate eigenstructures, e.g.,

$$\text{eigenvalues: } \lambda_1 = \lambda_2 = 0, \lambda_3, \dots, \lambda_n \text{ distinct}$$
$$\text{and } \mathrm{Re}(\lambda_3), \dots, \mathrm{Re}(\lambda_n) < 0$$
$$\underset{\text{(independent)}}{\text{eigenvectors: } [v_1\ v_2\ v_3 \cdots v_n]} = \begin{bmatrix} 1 & f_1 & 0 & \cdots & 0 \\ 1 & f_2 & 0 & \cdots & 0 \\ 1 & f_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_n & 0 & \cdots & 1 \end{bmatrix}. \tag{19}$$

2) Let f_1 and f_2 be the first two components of f, and set

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)(|x_2 - x_1|^2 - d^2|f_2 - f_1|^2) \\ (x_1 - x_2)(|x_1 - x_2|^2 - d^2|f_1 - f_2|^2) \end{bmatrix} =: r(x_1, x_2).$$

3) Set the control

$$u := Fx + B^{-1} \begin{bmatrix} r(x_1, x_2) \\ 0 \end{bmatrix}. \tag{20}$$

The idea of the above-mentioned synthesis procedure is to first use eigenstructure assignment to achieve a desired formation configuration with two leaders, and then control the size of the formation by stabilizing the distance between the two leaders to the prescribed d. The latter is inspired by the work in [13]. Our result is as follows.

Proposition 4: Consider the multiagent system (2) and let $f \in \mathbb{C}^n$, $d > 0$. Then, the control u in (20) synthesized by the fixed-size formation synthesis procedure solves Problem 2 for all initial conditions $x(0)$ with $x_1(0) \neq x_2(0)$.

Proof: First, by a similar argument to that in the proof of Theorem 1, we can show that the desired eigenvalues/eigenvectors (two eigenvalues at 0 with eigenvectors $\mathbf{1}$ and f; all other eigenvalues with negative real parts) may always be assigned for the multiagent system (2). As a result

$$(\forall x(0) \in \mathbb{C}^n)(\exists c, c' \in \mathbb{C}) \lim_{t\to\infty} x(t) = c\mathbf{1} + c' f.$$

Moreover, choosing the eigenstructure in (19) and following similarly to Proposition 1, we can show that the resulting topology defined by F is 2-rooted with nodes 1 and 2 the only two roots.

With the 2-rooted topology and the design in Step ii), it follows from [13, Th. 4.4] that for all $x(0)$ with $x_1(0) \neq x_2(0)$, we have $c' = dfe^{j\theta}$ for some $\theta \in [0, 2\pi)$. ■

An illustrative example of achieving fixed-size formations is provided in Section VII below.

B. Circular Motion

We apply the eigenstructure assignment approach to solve a cooperative circular motion problem, in which all the agents circle around the same center while keeping a desired formation configuration. This cooperative task may find useful applications in target tracking and encircling (e.g., [27] and [28]).

Problem 3: Consider the multiagent system (2) and specify $f \in \mathbb{C}^n$ $(f \neq 0)$ and $b \in \mathbb{R}$ $(b \neq 0)$. Design a state feedback control $u = Fx$ such that for every initial condition $x(0)$, $\lim_{t\to\infty} x(t) = c\mathbf{1} + c' f e^{bjt}$ for some $c, c' \in \mathbb{C}$ and $j = \sqrt{-1}$.

In Problem 3, the goal is that all the agents of (2) circle around the same center c at rate b, while keeping the formation configuration f scaled by $|c'|$.

Our result is the following.

Proposition 5: Consider the multiagent system (2) and let $f \in \mathbb{C}^n$, $b \in \mathbb{R}$. Then, there always exists a state feedback control $u = Fx$ that solves Problem 3.

Proof: By a similar argument to that in the proof of Theorem 1, we can show that for (2) there always exists F such that $(A + BF)$ has the following eigenstructure:

$$\text{eigenvalues: } \lambda_1 = 0, \lambda_2 = bj, \lambda_3, \dots, \lambda_n \text{ distinct}$$
$$\text{and } \mathrm{Re}(\lambda_3), \dots, \mathrm{Re}(\lambda_n) < 0$$
$$\text{eigenvectors: } v_1 = \mathbf{1}, v_2 = f, \{v_1, v_2, \dots, v_n\} \text{ independent.}$$

Hence

$$(\forall x(0) \in \mathbb{C}^n)(\exists c, c' \in \mathbb{C}) \lim_{t\to\infty} x(t) = c\mathbf{1} + c' f e^{bjt}.$$

That is, Problem 3 is solved. ■

The key point to achieving circular motion is to assign one, and only one, pure imaginary eigenvalue bj, associated with the formation vector f. The circular motion is counterclockwise if $b > 0$, and clockwise if $b < 0$. One may easily speed up or slow down the circular motion by specifying the value $|b|$.

Also note that, by a similar synthesis procedure to that for fixed-size formation in the previous section, the multiagent system (2) can be made to achieve circular motion while keeping a fixed-size formation with some specified size $d > 0$.

VII. Simulations

We illustrate the eigenstructure assignment based approach by simulation examples. For all the examples, we consider the multiagent system (2) with five heterogeneous agents, where

$$A = \mathrm{diag}(1.6, 4.7, 3.0, -0.7, -4.2)$$
$$B = \mathrm{diag}(0.2, 1.5, -0.5, -3.3, -3.7).$$

Thus, the first three agents are unstable, whereas the latter two are stable; all agents are stabilizable.

First, to achieve a scalable (regular) pentagon formation, assign the following eigenstructure:

$$\text{eigenvalues: } \{\lambda_1, \dots, \lambda_5\} = \{0, -1, -2, -3, -4\}$$
$$\text{eigenvectors: } [v_1 \cdots v_5] = \begin{bmatrix} e^{\frac{2\pi j\times 1}{5}} & -1 & 0 & 0 & 0 \\ e^{\frac{2\pi j\times 2}{5}} & 1 & 0 & 0 & 0 \\ e^{\frac{2\pi j\times 3}{5}} & -2 & -1 & 1 & 0 \\ e^{\frac{2\pi j\times 4}{5}} & -2 & 0 & -1 & 0 \\ e^{\frac{2\pi j\times 5}{5}} & -2 & 0 & -2 & 1 \end{bmatrix}.$$

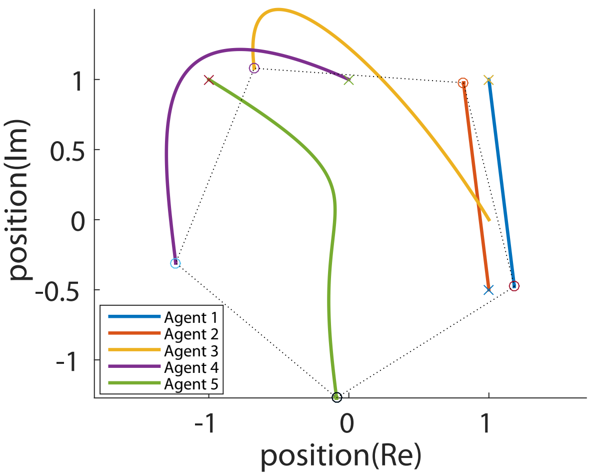


Fig. 3. Scalable regular pentagon formation (x: initial positions, ○: steady-state positions).

By (7), we compute the following control gain matrix:

$$F = \begin{bmatrix} -10.5 - 1.8164j & 2.5 - 1.8164j & 0 & 0 & 0 \\ 0.3333 + 0.2422j & -3.4667 + 0.2422j & 0 & 0 & 0 \\ -0.4721 - 2.3511j & -0.4721 - 2.3511j & 10 & -2 & 0 \\ -0.0442 - 0.4403j & 1.1679 - 0.4403j & 0 & 0.697 & 0 \\ -0.3979 + 0.4391j & 0.1427 + 0.4391j & 0 & -0.5405 & -0.0541 \end{bmatrix}.$$

Simulating the closed-loop system with initial condition $x(0) = [1+j \ \ 1-0.5j \ \ 1 \ \ j \ \ -1+j]^\top$, the result is displayed in Fig. 3. Observe that a regular pentagon is formed, and the topology determined by F contains a spanning tree.

Next, to achieve a fixed-size pentagon formation, we follow the method presented in Section VI-A. First, assign the following eigenstructure:

$$\text{eigenvalues: } \{\lambda_1, \ldots, \lambda_5\} = \{0, 0, -1, -2, -3\}$$

$$\text{eigenvectors: } [v_1 \cdots v_5] = \begin{bmatrix} 1 & e^{\frac{2\pi j \times 1}{5}} & 0 & 0 & 0 \\ 1 & e^{\frac{2\pi j \times 2}{5}} & 0 & 0 & 0 \\ 1 & e^{\frac{2\pi j \times 3}{5}} & 1 & 0 & 0 \\ 1 & e^{\frac{2\pi j \times 4}{5}} & 0 & 1 & 0 \\ 1 & e^{\frac{2\pi j \times 5}{5}} & 0 & 0 & 1 \end{bmatrix}$$

and by (7), compute the control gain matrix, as given by

$$F = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 \\ 0 & -3.1333 & 0 & 0 & 0 \\ 0.618 + 1.9021j & -2.618 - 1.9021j & 8 & 0 & 0 \\ -0.303 + 0.9326j & -0.303 - 0.9326j & 0 & 0.3939 & 0 \\ -1.0614 + 0.7711j & 0.2506 - 0.7711j & 0 & 0 & -0.3243 \end{bmatrix}.$$

Thus, the topology determined by F is 2-rooted with nodes 1 and 2 the only two roots. Then, for different sizes ($d = 5, 10, 15$), we obtain by (20) the control u. Simulating the closed-loop system with the same initial condition $x(0)$ as previously mentioned, the result is displayed in Fig. 4, where pentagons with specified sizes are formed.

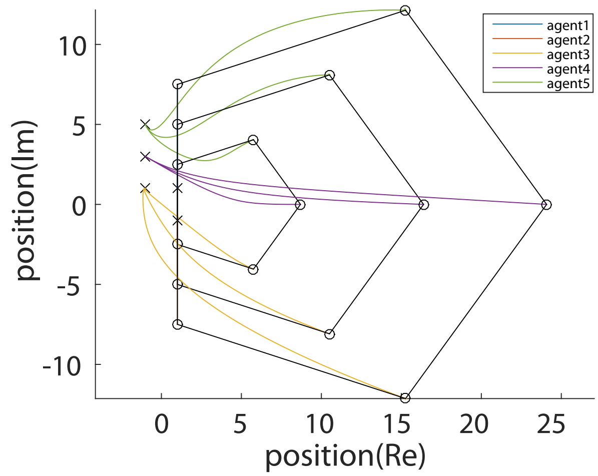


Fig. 4. Fixed-size regular pentagon formation, with size $d = 5, 10, 15$ (x: initial positions, ○: steady-state positions).

TABLE I
COMPARISON OF COMPUTATION TIME (UNIT: SECONDS)

agent #	centralized method by (7)	hierarchical method in Sec. V
100	0.398	0.027
500	57.308	0.179
900	552.8419	0.394
1000	1068.729	0.525

Finally, we present an empirical study on the computation time of synthesizing feedback matrix F. In particular, we compare the centralized synthesis by (7) and the hierarchical synthesis in Section V; the result is listed in Table I for different numbers of agents.[4] Here, for the hierarchical synthesis, we partition the agents in such a way that the number of groups and the number of agents in each group are "balanced" (to make $\tilde{n}$ small): e.g., 100 agents are partitioned into 10 groups of 10 agents each; 500 agents are partitioned into 16 groups of 23 agents each *plus* 6 groups of 22 each. Observe that the hierarchical synthesis is significantly more efficient than the centralized one, and the efficiency increases as the number of agents increases. In particular, for 1000 agents only 0.525 s needed, the hierarchical approach might well be sufficient for many practical purposes.

VIII. CONCLUDING REMARKS

We have proposed a top-down, eigenstructure assignment based approach to synthesize state feedback control for solving multiagent formation problems. The relation between the eigenstructures used in control synthesis and the resulting topologies among agents have been characterized, and special topologies have been designed by choosing appropriate eigenstructures. More general cases where the initial interagent coupling is arbitrary and/or there exist nonstabilizable agents have been studied, and a hierarchical synthesis procedure has been presented that improves computational efficiency. Furthermore, the approach has been extended to achieve fixed-size formation and circular motion.

[4]Computation is done by MATLAB R2014b on a laptop with Intel(R) Core(TM) i7-4510U CPU@2.00 GHz 2.60 GHz and 8.00 GB memory.

In our view, the proposed top-down approach to multiagent formation control is *complimentary* to the existing (mainstream) bottom-up approach (rather than *opposed* to). Indeed, the bottom-up approach, if successful, can produce scalable control strategies effective for possibly time-varying topologies, nonlinear agent dynamics, and robustness issues such as communication failures, which are the cases very difficult to be dealt with by the top-down approach. On the other hand, bottom-up design is generally challenging, requiring significant insight into the problem at hand and possibly many trials and errors in the design process; by contrast, top-down design is straightforward and can be automated by algorithms. Hence, we suggest the following. When a control researcher or engineer faces a distributed control design problem for achieving some new cooperative tasks, one can start with a linear time-invariant version of the problem and try the top-down approach to derive a solution. With the ideas and insights gained from such a solution, one may then try the bottom-up design possibly for time-varying and nonlinear cases.

In future work, we aim to apply the top-down, eigenstructure assignment based approach to solve more complex cooperative control problems for higher order multiagent systems. Our immediate goal is to achieve formations in three dimensions (by treating the state of each agent as a three-dimensional real vector), as well as to track moving target formations (by introducing leaders and velocity feedback). In addition, since our current solution generally relies on the availability of agents' position measurement (except for the case of consensus where f is the vector of all ones), we aim to address the case where only *relative* position measurement is available.

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Takatoshi Motoyama received the B.Eng. degree in 2016 and the M.Eng. degree in electrical and information engineering from Osaka City University, Osaka, Japan, in 2018. His research interest is multi-agent formation control.

Kai Cai (S'08–M'12–SM'17) received the B.Eng. degree in electrical engineering from Zhejiang University, Hangzhou, China, in 2006, the M.A.Sc. degree in electrical and computer engineering from the University of Toronto, Toronto, ON, Canada, in 2008, and the Ph.D. degree in systems science from the Tokyo Institute of Technology, Tokyo, Japan, in 2011.

He is currently an Associate Professor with Osaka City University, Osaka, Japan; preceding this position, he was an Assistant Professor from 2013 to 2014 and a Postdoctoral Fellow from 2011 to 2013 with the University of Tokyo. He is the coauthor (with W. M. Wonham) of the *Supervisory Control of Discrete-Event Systems* (Springer, 2018) and *Supervisor Localization* (Springer, 2015). His research interests include distributed control of discrete-event systems and cooperative control of networked multiagent systems.

Dr. Cai was the recipient of the Best Paper Award of SCIE in 2013, the Best Student Paper Award of the IEEE Multi-Conference on Systems and Control, and the Young Author's Award of SCIE in 2010.