Transfer Function

Review: Laplace transform

Consider a continuous-time, real-valued function $f(t)$, where $-\infty < t < \infty$.

Laplace transform of $f(t)$ is:

$$F(s) := \int_0^\infty f(t) e^{-st} dt$$

where $s \in \mathbb{C}$ is a complex variable.

Table

$f(t)$	$F(s)$	
$1_+(t)$	$\frac{1}{s}$	unit step
e^{at}	$\frac{1}{s-a}$	
$\dot{f}(t)$	$sF(s) - f(0)$	valid if $f(t)$ is differentiable at $t=0$

Recall state model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x(t) \in \mathbb{R}^n$: state vector
$u(t) \in \mathbb{R}^m$: input vector
$y(t) \in \mathbb{R}^p$: output vector
A, B, C, D are constant matrices

We are going to take Laplace transforms of these two equations

Laplace transform of vector signals

$$\text{For } x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \text{ define } X(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

$$\text{For } u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \text{ define } U(s) = \begin{bmatrix} U_1(s) \\ \vdots \\ U_m(s) \end{bmatrix}$$

$$\text{For } y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}, \text{ define } Y(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_p(s) \end{bmatrix}$$

Laplace transform of vector signals

$$\text{derivative } \dot{x}(t) \longrightarrow sX(s) - x(0)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s) \quad (x(0) = 0)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

$$y(t) = Cx(t) + Du(t) \longrightarrow Y(s) = CX(s) + DU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$Y(s) = (C(sI - A)^{-1}B + D)U(s)$$

Transfer function model

The *transfer funciton* of

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is the function $G(s)$ satisfying $Y(s) = G(s)U(s)$ with $x(0) = 0$, and is given by

$$G(s) := C(sI - A)^{-1}B + D$$

$$(G(s) \text{ is a } p \times m \text{ matrix})$$

Transfer function model

Single input, single output case: $G(s)$ is 1×1

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \frac{1}{\det(sI-A)} C\,\mathrm{adj}(sI - A)B + D$$

$$= \frac{C\,\mathrm{adj}(sI-A)B + D\det(sI-A)}{\det(sI-A)}$$

$$= \frac{N(s)}{D(s)}$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

$$\text{Compute } G(s) = C(sI - A)^{-1}B + D$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

$G(s) = \frac{1}{s^2-1}$

Suppose input $u(t) = 1$ for $t \geq 0$ (unit step)

Let's find output $y(t)$

$$Y(s) = G(s)U(s) = \frac{1}{s^2-1}\frac{1}{s} = \frac{1}{s(s+1)(s-1)}$$
$$= -\frac{1}{s} + \frac{0.5}{s+1} + \frac{0.5}{s-1}$$

So $y(t) = -1 + \frac{1}{2}\mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{t}$

Example

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = 0$$

Compute $G(s) = C(sI - A)^{-1}B + D$

Example

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = 0$$

Compute $G(s) = C(sI - A)^{-1}B + D$

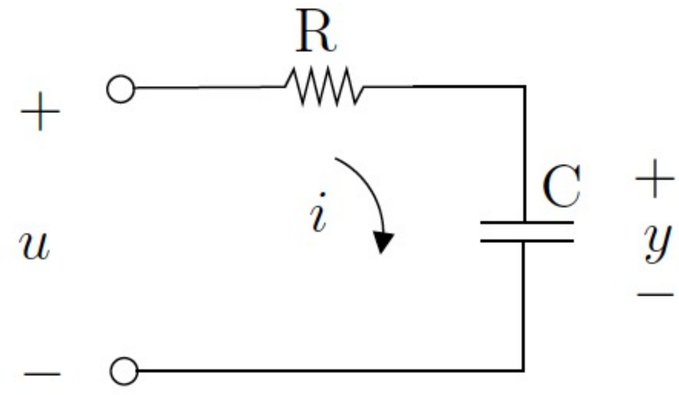
$$= \frac{1}{s^2(s^2+1)} \begin{bmatrix} s^2+1 & 1 \\ 1 & s^2+1 \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Transfer function model

For some simple systems, we can get transfer function model directly without first getting state model

Ex. RC filter



$$-u + Ri + y = 0,\ i = C\frac{dy}{dt}$$

$$RC\dot{y} + y = u$$

$$RCsY(s) + Y(s) = U(s) \qquad (y(0) = 0)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{RCs+1}$$

Transfer function model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{RCs+1}$$

Note: unit of RC is second, called *time constant* of the circuit

Note: pole of $G(s)$ at $s = -\frac{1}{RC}$;
smaller time constant implies farther left the pole

Note: the *DC gain* of the circuit is $G(0) = 1$;
if $u(t)$ is a constant voltage, then in steady state $y(t) = u(t)$

Note: this is a *lowpass* circuit

Example

$G(s) = 2$: pure gain

$G(s) = \frac{1}{s}$: single integrator; $y(t) = \int_{-\infty}^{t} u(\tau) d\tau$

$G(s) = \frac{1}{s^2}$: double integrator

$G(s) = s$: differentiator; $y(t) = \dot{u}(t)$
(at best an approximation to a real system)

$G(s) = \mathrm{e}^{-\tau s}$ $(\tau > 0)$: time-delay system; not rational

$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$: standard second-order transfer function
($\omega_n > 0$: natural frequency; $\zeta \geq 0$: damping constant)

Example

$G(s) = K_1 + \frac{K_2}{s} + K_3 s$: proportional-integral-derivative (PID) controller

Note: $G(s) = \frac{K_3 s^2 + K_1 s + K_2}{s}$ is improper

Note: $K_3 s$ is a differentiator; at best an approximation to a real system; a better approximation:
$G(s) = K_1 + \frac{K_2}{s} + \frac{K_3 s}{\varepsilon s + 1}$, where $\varepsilon > 0$ is a small positive number

Realization

Inverse problem: given a transfer function $G(s)$,
find a state model A, B, C, D s.t. $G(s) = C(sI - A)^{-1}B + D$

This state model A, B, C, D is called a *realization* of $G(s)$

Note: each $G(s)$ has an infinite number of state realizations

Note: every proper, rational $G(s)$ has a state realization

Example

$G(s) = \frac{1}{2s^2 - s + 3} = \frac{Y(s)}{U(s)}$

$(2s^2 - s + 3)Y(s) = U(s)$

$2s^2Y(s) - sY(s) + 3Y(s) = U(s)$

$2\ddot{y} - \dot{y} + 3y = u$

Taking $x_1 = y$, $x_2 = \dot{y}$, we get

$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u$

$y = x_1$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This technique extends to

$$G(s) = \frac{\text{constant}}{\text{polynomial of degree } n}$$

Example

$G(s) = \frac{s-2}{2s^2-s+3} = \frac{Y(s)}{U(s)}$

Introduce an auxiliary $V(s)$ s.t.
$Y(s) = (s-2)V(s)$, $V(s) = \frac{1}{2s^2-s+3}U(s)$

$y = \dot{v} - 2v$, $2\ddot{v} - \dot{v} + 3v = u$

Taking $x_1 = v$, $x_2 = \dot{v}$, we get

$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u$

$y = x_2 - 2x_1$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This technique extends to any strictly proper rational $G(s)$

Example

Proper rational $G(s) = \frac{s+1}{s}$ (not strictly proper)

i.e. $G(s) = \frac{N(s)}{D(s)}$, $N(s)$ and $D(s)$ have the same degree

Divide $N(s)$ by $D(s)$ to get $G(s) = c + G_1(s)$

where c is a constant, and $G_1(s)$ is strictly proper

In this case we get A, B, C to realize $G_1(s)$, and set $D = c$