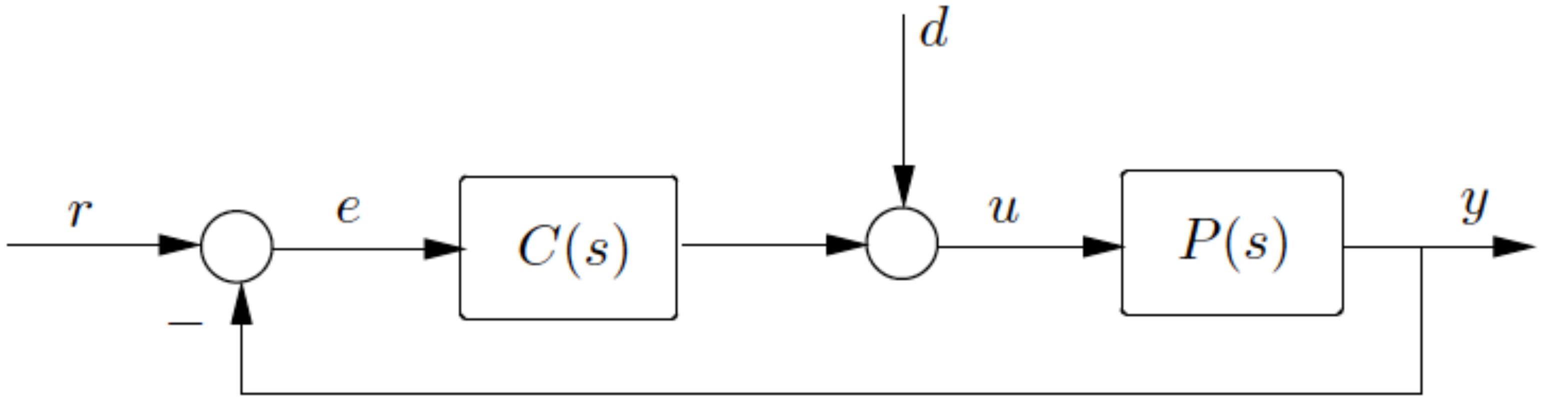
# Nyquist Criterion

# Intuition




Given a feedback loop, we want to analyze if it is stable

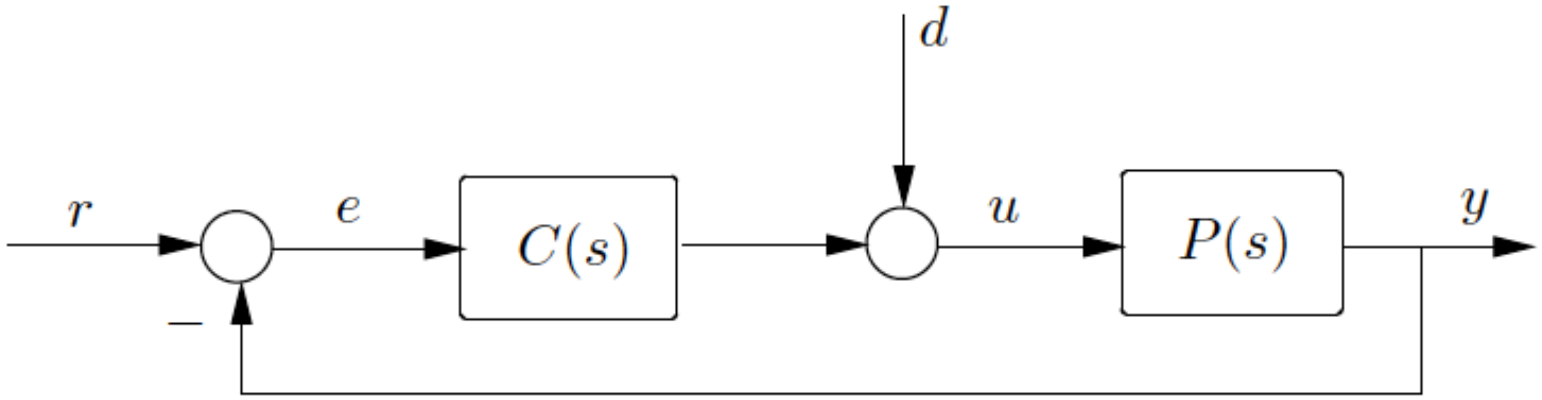
So far we have had two computational methods:

1) Compute eigenvalues of $A_{cl} = \begin{bmatrix} A_p & B_pC_c \\ -B_cC_p & A_c \end{bmatrix}$

2) Compute zeros of $D_pD_c + N_pN_c$

Nyquist criterion is a graphical method for this purpose

# Intuition




Plant $P(s) = \frac{N_p(s)}{D_p(s)}$ is rational and strictly proper,

controller $KC(s) = \frac{KN_c(s)}{D_c(s)}$ is rational and proper $(K \neq 0)$

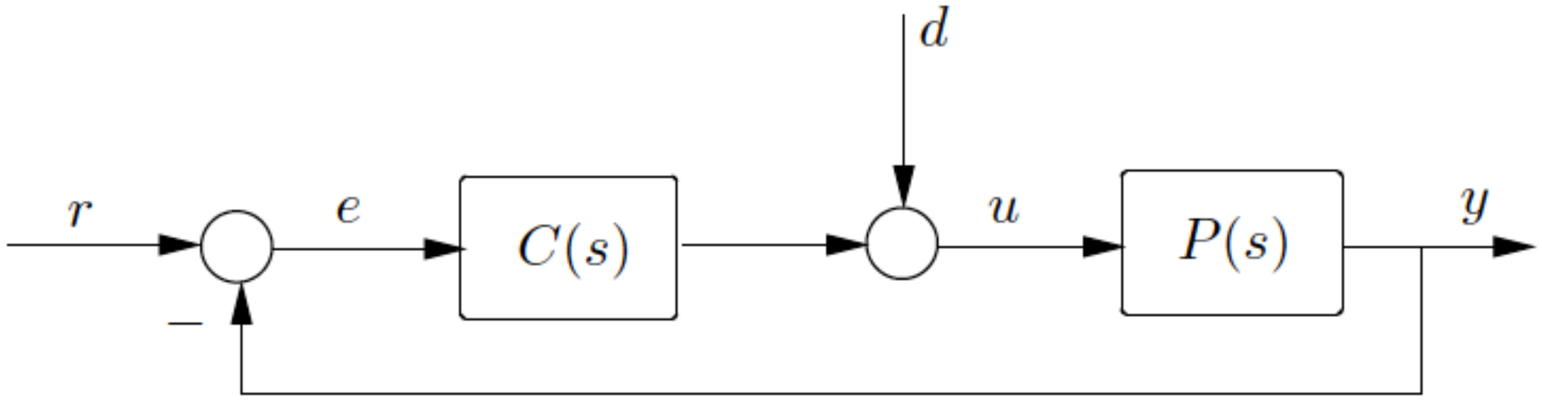
Closed-loop characteristic polynomial is $D_p(s)D_c(s) + KN_p(s)N_c(s)$

Note $1 + KP(s)C(s) =$

If there is no unstable pole-zero cancellation,
$D_pD_c + KN_pN_c$ and $1 + KPC$ have the same unstable zeros

Feedback stability $\Leftrightarrow$ $D_pD_c + KN_pN_c$ has no unstable zeros
$\Leftrightarrow$ $1 + KPC$ has no unstable zeros

# Intuition




Let $M(s) := 1 + KP(s)C(s)$

Then $P(s)C(s) = \frac{1}{K}M(s) - \frac{1}{K}$

Feedback stability $\Leftrightarrow M(s)$ has no zeros in $\mathrm{Re}(s) \geq 0$

$\Leftrightarrow P(s)C(s)$ has no zeros in $\mathrm{Re}(s) \geq -\frac{1}{K}$

Nyquist criterion is a graphical test for this condition

# Principle of the Argument

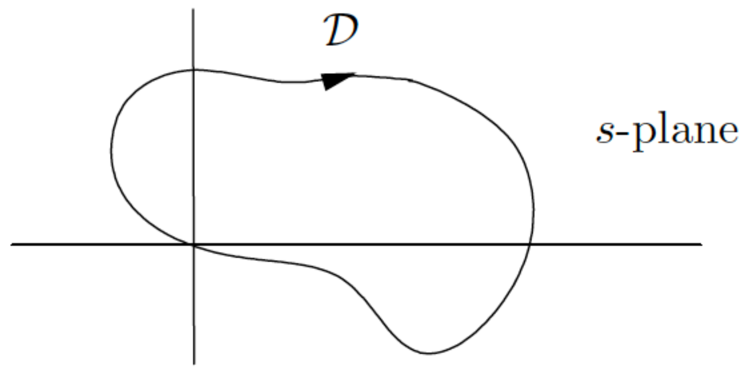
# Principle of the argument

Principle of the argument is from complex function theory

"Argument" refers to the angle of a complex number

Principle of the argument involves two things:
1) a curve in the complex plane
2) a transfer function

# Principle of the argument

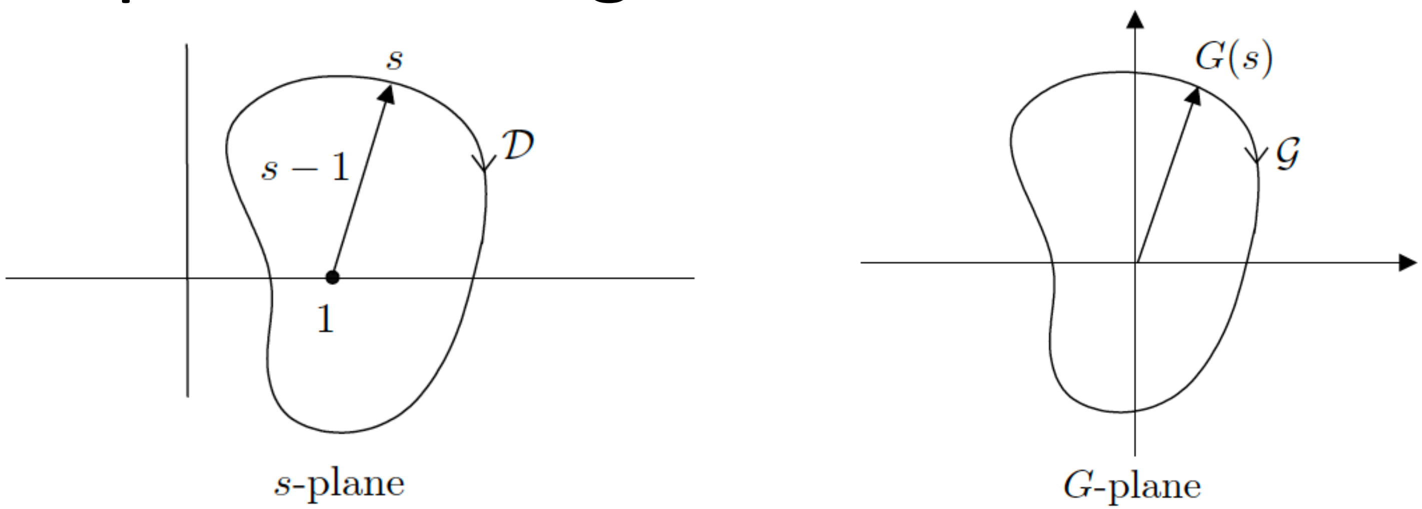



Consider a closed curve $\mathcal{D}$ (also called contour)
with no self-intersections and with clockwise (CW) orientation

Consider a rational transfer function $G(s)$,
having no zeros or poles on the curve $\mathcal{D}$

For every point $s$ in the complex plane, $G(s)$ is a point
in the complex plane, thus we can draw of graph for $G(s)$

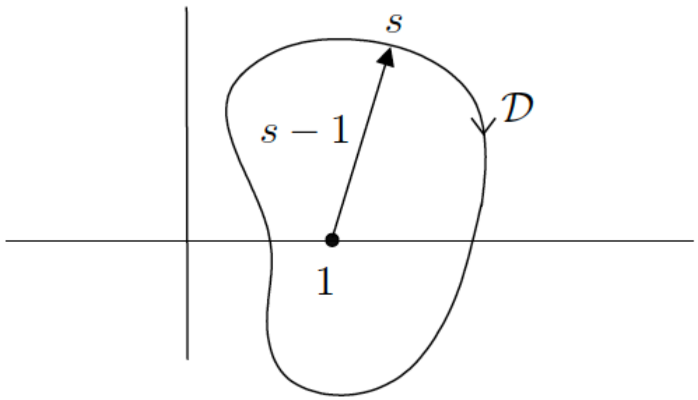
# Principle of the argument

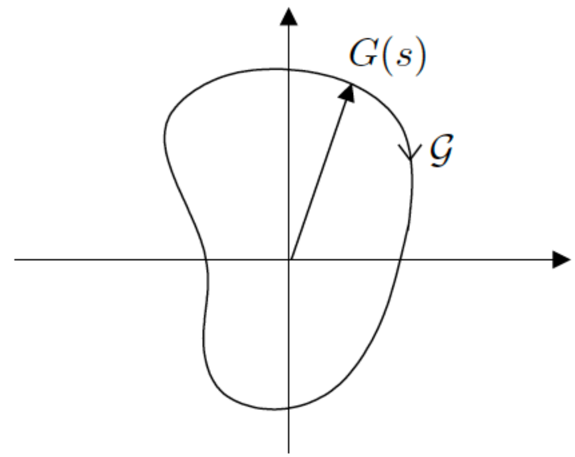



Draw two copies of the complex plane:
"$s$-plane" for $\mathcal{D}$ and "$G$-plane" for $G(s)$

As $s$ goes once around $\mathcal{D}$ from any starting point,
the point $G(s)$ traces out a closed curve $\mathcal{G}$
the image of $\mathcal{D}$ under $G(s)$

# Example



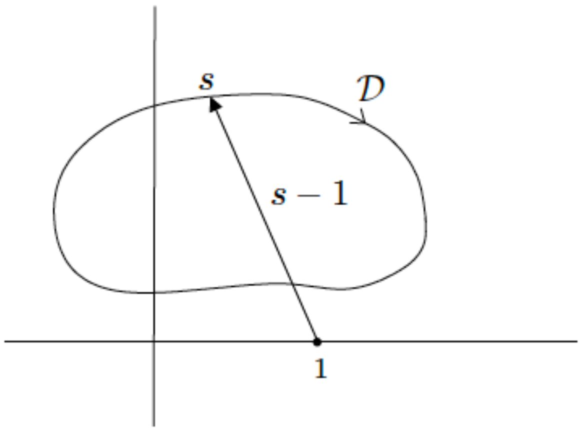



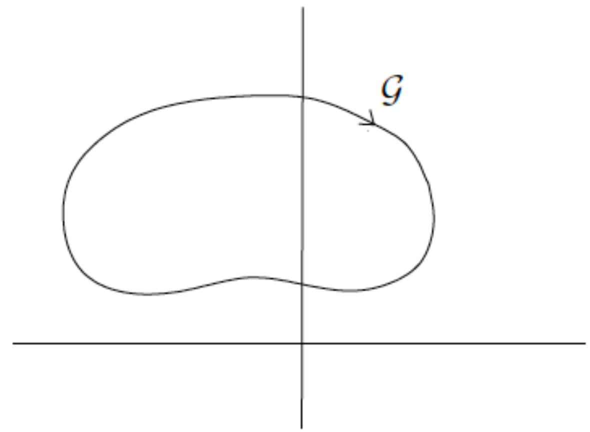

$$G(s) = s - 1$$

$\mathcal{G}$ is simply $\mathcal{D}$ shifted to the left by one unit

Since $\mathcal{D}$ encircles the zero of $G(s)$ at $s = 1$,
$\mathcal{G}$ encircles the origin ($s = 0$) once CW

# Example



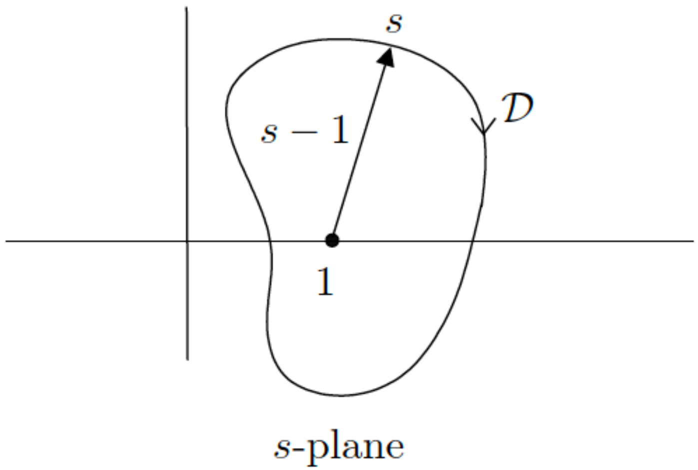




$G(s) = s - 1$

$\mathcal{G}$ is simply $\mathcal{D}$ shifted to the left by one unit

Since $\mathcal{D}$ does not encircle the zero of $G(s)$ at $s = 1$,
$\mathcal{G}$ does not encircle the origin

# Example

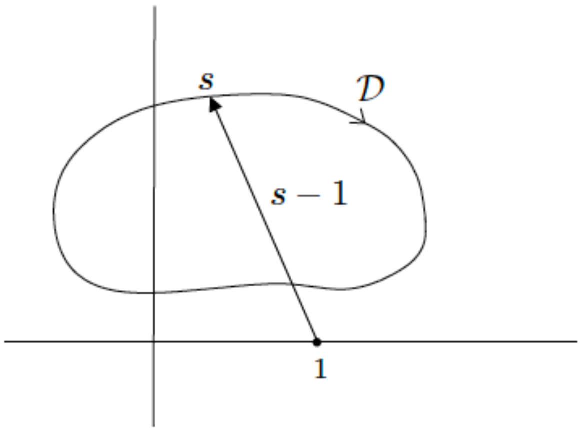



$G(s) = \frac{1}{s-1}$

The angle of $G(s)$ equals the negative of the angle of $s - 1$:
$\angle G(s) = \angle 1 - \angle(s - 1) = -\angle(s - 1)$

If $\mathcal{D}$ encircles the pole of $G(s)$ at $s = 1$,
$\mathcal{G}$ encircles the origin ($s = 0$) once counter-clockwise (CCW)

# Example

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Since $\mathcal{D}$ does not encircle the pole of $G(s)$ at $s=1$,
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# Principle of the argument

Relation between # poles and # zeros of $G(s)$ encircled by $\mathcal{D}$ and # times the origin encircled by $\mathcal{G}$

Suppose that $G(s)$ has no poles or zeros on $\mathcal{D}$.
If $\mathcal{D}$ encircles $n$ poles and $m$ zeros of $G(s)$,
then $\mathcal{G}$ encircles the origin exactly $n - m$ times CCW

# Principle of the argument

Proof: Write $G(s) = K \frac{\Pi_i (s-z_i)}{\Pi_i (s-p_i)}$

where $K$ is a real gain, $z_i$ are zeros, and $p_i$ are poles

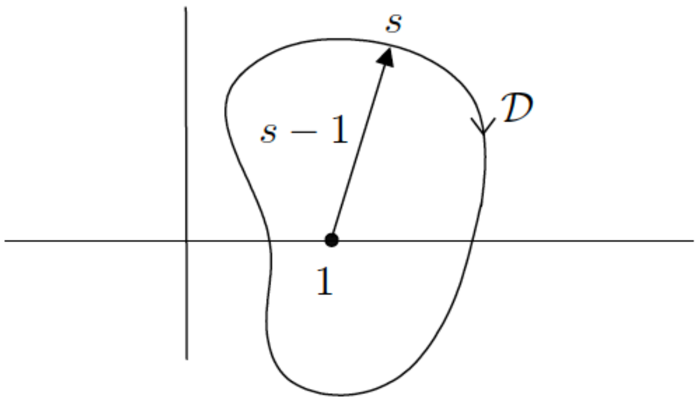
Then the angle of $G(s)$ is
$$\begin{aligned} \angle G(s) &= \angle K + \Sigma \angle (s - z_i) - \Sigma \angle (s - p_i) \\ &= \Sigma \angle (s - z_i) - \Sigma \angle (s - p_i) \end{aligned}$$

Since $\mathcal{D}$ encircles $m$ zeros $z_i$ and $n$ poles $p_i$

the net change in $\angle G(s)$ is $m(-2\pi) - n(-2\pi) = (n - m)2\pi$

Therefore $\mathcal{G}$ encircles the origin $n - m$ times CCW

# Examples




$s$-plane

$$G(s) = \frac{s}{s-1}$$

$$G(s) = \frac{s-1}{s(s-10)}$$

$$G(s) = \frac{s-1}{s(s-1.1)(s-10)}$$