Laplace transform

Roadmap

ODE (physical laws)

State model

Transfer function model

Laplace transform

Consider a continuous-time, real-valued function $f(t)$, where $-\infty < t < \infty$.

Laplace transform of $f(t)$ is:

$$F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

where $s \in \mathbb{C}$ is a complex variable.

Note: the integral is one-sided: ignores $f(t)$ for $t < 0$

Question: does the integral converge?

Laplace transform

Assumption 1: $f(t)$ is *piecewise continuous* for $t \geq 0$.

(f is continous except possibly at a countable number of times $0 = t_0 < t_1 < \cdots$, where the intervals be lower bounded)

Ex. sinusoid

Ex. square wave

Ex. $f(t) = 0$ if t is rational; $f(t) = 1$ if t is not rational

Assumption 1 ensures for every finite T:
(Riemann integral) $\int_0^T f(t)\mathrm{e}^{-st}dt$ exists

Laplace transform

Assumption 2: $f(t)$ is *exponentially bounded* for $t \geq 0$.

(there exist $M \geq 0$ and $a \in \mathbb{R}$ such that $|f(t)| \leq M\mathrm{e}^{at}$)

Ex. $f(t) = \mathrm{e}^{t}$

Ex. $f(t) = \mathrm{e}^{(t^2)}$

Ex. $f(t) = \frac{1}{t-1}$ if $t \neq 1$; $f(t) = 0$ if $t = 1$

Assumption 2 ensures if $\mathrm{Re}(s)$ is sufficient large:
$\lim_{T\to\infty} \int_0^T f(t)\mathrm{e}^{-st}dt$ exists

Laplace transform

Let $\mathrm{Re}(s)$ be sufficient large.

To prove $\lim_{T\to\infty}\int_0^T f(t)\mathrm{e}^{-st}dt$ exists,

it suffices to prove $\lim_{T\to\infty}\int_0^T |f(t)\mathrm{e}^{-st}|dt$ exists

$$\begin{aligned}
\int_0^T |f(t)\mathrm{e}^{-st}|dt &\le M\int_0^T \mathrm{e}^{at}|\mathrm{e}^{-st}|dt \\
&= M\int_0^T \mathrm{e}^{at}\mathrm{e}^{-\mathrm{Re}(s)t}dt \\
&= M\int_0^T \mathrm{e}^{-(\mathrm{Re}(s)-a)t}dt \\
&= \tfrac{M}{\mathrm{Re}(s)-a}(1-\mathrm{e}^{-(\mathrm{Re}(s)-a)T}) \\
&\le \tfrac{M}{\mathrm{Re}(s)-a} \qquad (\text{for } \mathrm{Re}(s) > a)
\end{aligned}$$

Therefore $\lim_{T\to\infty}\int_0^T |f(t)\mathrm{e}^{-st}|dt$ exists.

Recap

Consider a continuous-time, real-valued function $f(t)$, where $-\infty < t < \infty$.

Laplace transform of $f(t)$ is:

$$F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}dt$$

where $s \in \mathbb{C}$ is a complex variable.

Assumption 1: $f(t)$ is *piecewise continuous* for $t \geq 0$.

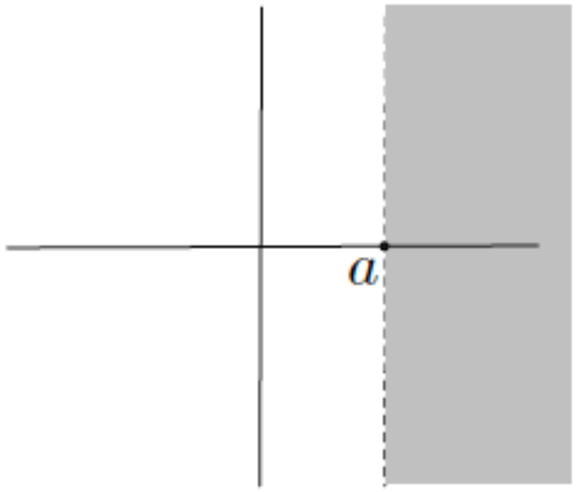
Assumption 2: $f(t)$ is *exponentially bounded* for $t \geq 0$.
i.e. $|f(t)| \leq M\mathrm{e}^{at}$

$F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}dt$ exists if $\mathrm{Re}(s) > a$

Recap

$F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}dt$ exists if $\mathrm{Re}(s) > a$

"region of convergence (ROC)": open right half-plane



Example

unit step: $f(t) = 1$ for $t \geq 0$

ROC

"region of convergence (ROC)": open right half-plane

Note: no poles of $F(s)$ inside ROC

Note: a pole of $F(s)$ on the boundary of ROC
(if ROC is not the entire complex plane)

Note: given $F(s)$ to find ROC, locate all poles and draw a vertical dashed line through the pole(s) farthest to the right

ex. $F(s) = \frac{1}{(s^2+1)(2s-1)}$

Example

blowing-up exponential: $f(t) = \mathrm{e}^{2t}$

Example

sinusoid: $f(t) = \sin(\omega t)$

Example

derivative: $\dot{f}(t) = \frac{df}{dt}$

Table

$f(t)$	$F(s)$	
$1_+(t)$	$\frac{1}{s}$	unit step
e^{at}	$\frac{1}{s-a}$	
$\dot{f}(t)$	$sF(s) - f(0)$	valid if $f(t)$ is differentiable at $t = 0$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	linearity
$f(t) * g(t)$	$F(s)G(s)$	convolution
t^n	$\frac{n!}{s^{n+1}}$	
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$	
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$	
$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	
$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$	