Stability

Intuition

Drinking a hot coffee in a car

Heartrate after playing soccer

Cancer cells' rampant growth

Epidemics

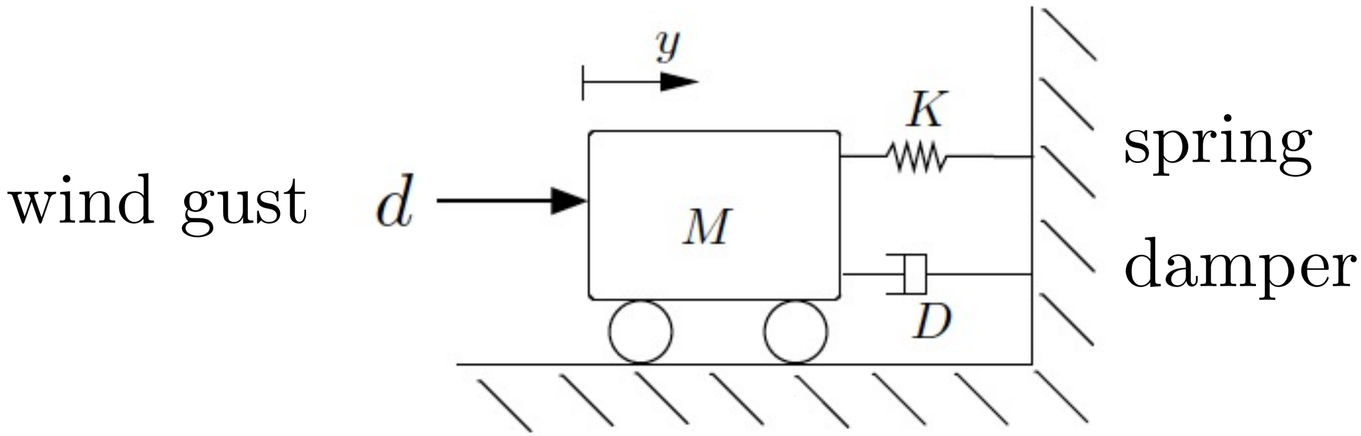
Intuition

A system is stable if it has the following two properties:

It returns to equilibrium after a perturbation of its state

It can accommodate a persistent disturbance

Example



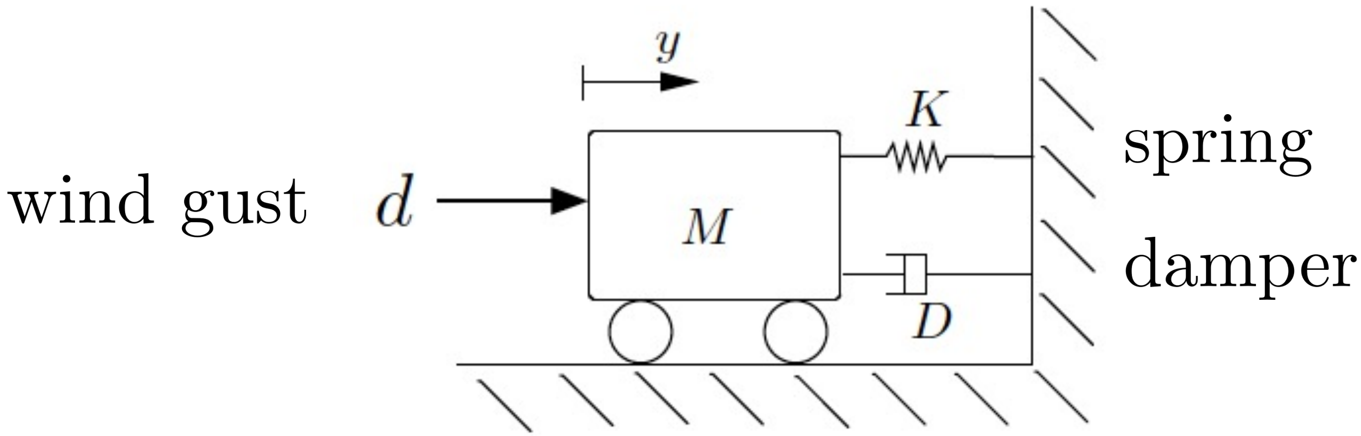
$$M\ddot{y} = d - Ky - D\dot{y}$$

$$\ddot{y} = d - y - \dot{y} \ (M = K = D = 1)$$

Taking $x_1 = y$, $x_2 = \dot{y}$, we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - x_1 + d$$
$$y = x_1$$

Example



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If $d(t) = 0$, will $x(t)$ converge to 0 for every $x(0)$?

If $d(t)$ is bounded, will $y(t)$ be bounded too?

Example

If $d(t) = 0$, will $x(t)$ converge to 0 for every $x(0)$?

$$\dot{x} = Ax$$

$$sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x(0)$$

$$X(s) = (sI - A)^{-1}x(0)$$

$$= \frac{1}{\det(sI - A)}\mathrm{adj}(sI - A)x(0)$$

$$= \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} x(0)$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Example

If $d(t) = 0$, will $x(t)$ converge to 0 for every $x(0)$?

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \frac{1}{s^2+s+1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$X_1(s) = \frac{(s+1)x_1(0)+x_2(0)}{s^2+s+1}, \; X_2(s) = \frac{-x_1(0)+sx_2(0)}{s^2+s+1}$$

Since the poles (i.e. $s^2 + s + 1 = 0$) have negative real part, for every $x(0)$, the final value of $x(t)$ is 0

Example

If $d(t)$ is bounded, will $y(t)$ be bounded too?

$$\dot{x} = Ax + Ed,\ y = Cx$$

$$sX(s) = AX(s) + ED(s),\ Y(s) = CX(s) \qquad (x(0) = 0)$$

$$\begin{aligned} Y(s) &= C(sI - A)^{-1}ED(s) \\ &= \tfrac{1}{\det(sI-A)} C\,\mathrm{adj}(sI - A)ED(s) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \tfrac{1}{s^2+s+1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} D(s) \\ &= \tfrac{1}{s^2+s+1} D(s) \end{aligned}$$

Since the poles (i.e. $s^2 + s + 1 = 0$) have negative real part, if $d(t)$ is bounded then $y(t)$ is bounded too

Definition of stability

For the system modeled by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

it is *stable* if the following two conditions hold:

1) with $u = 0$ and $x(0)$ arbitrary, the final value of $x(t)$ is 0
(asymptotic stability)

2) with $u(t)$ bounded and $x(0) = 0$, $y(t)$ is bounded
(bounded-input bounded-output stability)

Conditions for stability

1) with $u = 0$ and $x(0)$ arbitrary, the final value of $x(t)$ is 0
if and only if the zeros of the polynomial $\det(sI - A)$
all have negative real parts

2) with $u(t)$ bounded and $x(0) = 0$, $y(t)$ is bounded
if the zeros of the polynomial $\det(sI - A)$
all have negative real parts

Polynomial $\det(sI - A)$ is the *characteristic polynomial* of A

Zeros of $\det(sI - A)$ are the *eigenvalues* of A

Conditions for stability

The system modeled by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is stable if

the eigenvalues of A all have negative real parts

Examples

System $\dot{x} = -x$ is stable

System $\dot{x} = 0$ is unstable

System $\dot{x} = x$ is unstable

Examples

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Examples

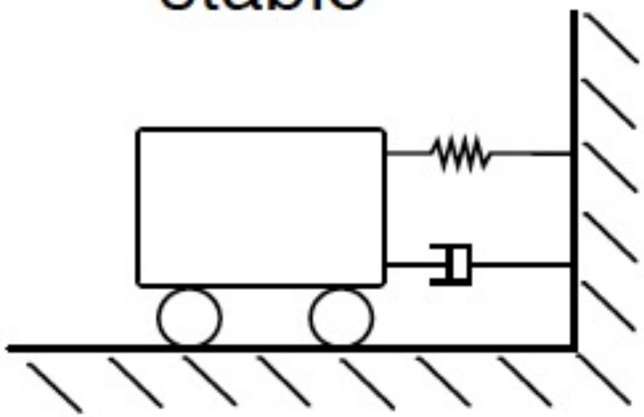
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

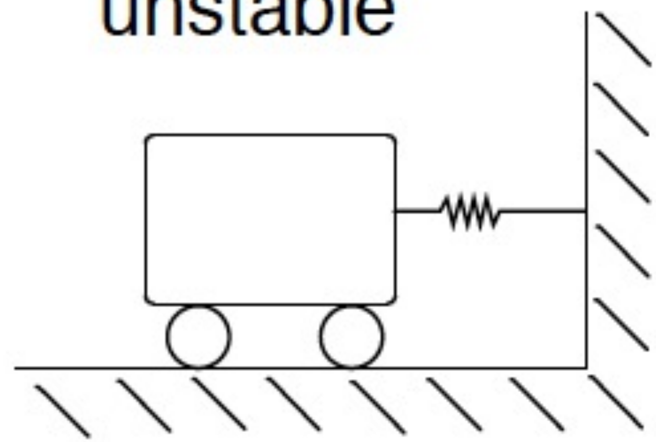
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

Examples

stable

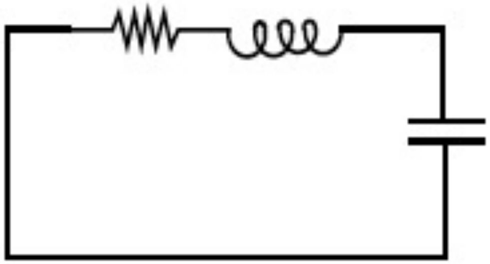


unstable

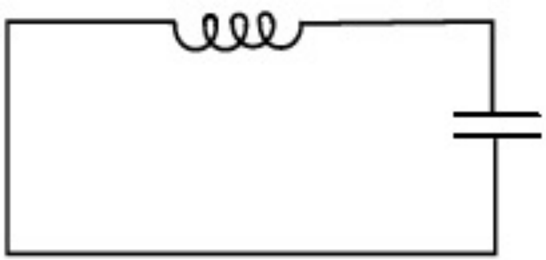


Examples

stable



unstable



Routh-Hurwitz criterion

Calculating the eigenvalues of A by hand may not be easy; use Matlab function eig(A)

For simple cases, it is useful to have an easy test for stability.

Routh-Hurwitz criterion is such a test (of historical interest)

Consider the characteristic polynomial of A is

$$p_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Routh-Hurwitz criterion is an algebraic test for if the zeros of $p_A(s)$ have negative real parts without actually calculating the zeros

Routh-Hurwitz criterion

$n = 1$: $p_A(s) = s + a_0$

Zero of $p_A(s)$ has negative real part iff $a_0 > 0$

$n = 2$: $p_A(s) = s^2 + a_1 s + a_0$

Zero of $p_A(s)$ has negative real part iff $a_0, a_1 > 0$

$n = 3$: $p_A(s) = s^3 + a_2 s^2 + a_1 s + a_0$

Zero of $p_A(s)$ has negative real part iff $a_0, a_1, a_2 > 0$
and $a_1 a_2 > a_0$