Undecidability

Undecidable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Undecidable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\mathrm{TM}} = \{< \mathbf{M}, s > | \ \mathbf{M} \text{ accepts } s\}$

Claim: cannot find a TM that decides A_{TM}

Undecidable problem

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Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

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Claim: cannot find a TM that decides A_{TM}

Note: A_{TM} is Turing-recognizable

$\mathbf{U} =$ "On input string $< \mathbf{M}, s >$: (universal TM)

1) Run s on $\mathbf{M}$ from the initial state q_0

2) If the run ends at q_a, accept; if ends at q_r, reject"

($\mathbf{U}$ loops on $< \mathbf{M}, s >$ if $\mathbf{M}$ loops on s; so $\mathbf{U}$ does not decide A_{TM})

Undecidable problem

Let's prove $A_{\mathrm{TM}} = \{< \mathbf{M}, s >| \ \mathbf{M} \text{ accepts } s\}$ is undecidable.

Assume A_{TM} is decidable (to obtain a contradiction).

Undecidable problem

Let's prove $A_{\text{TM}} = \{< \mathbf{M}, s >| \mathbf{M} \text{ accepts } s\}$ is undecidable.

Assume A_{TM} is decidable (to obtain a contradiction).

Then there is a decider $\mathbf{H}$ for A_{TM} (always halts):

$\mathbf{H} =$ "On input string $< \mathbf{M}, s >$:

1) Run s on $\mathbf{M}$ from the initial state q_0

2) If the run ends at q_a, accept; otherwise, reject"

$$\mathbf{H}(< \mathbf{M}, s >) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ accepts } s \\ \text{reject} & \text{if } \mathbf{M} \text{ does not accepts } s \end{cases}$$

Undecidable problem

Next consider a new decider $\mathbf{D}$ that uses $\mathbf{H}$ as subprocedure:

$\mathbf{D}$ = "On input string $< \mathbf{M} >$:

1) Run $\mathbf{H}$ on input string $< \mathbf{M}, < \mathbf{M} >>$

2) If $\mathbf{H}$ rejects, accept; $\mathbf{H}$ accepts, reject"

$$\mathbf{D}(< \mathbf{M} >) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ does not accepts } < \mathbf{M} > \\ \text{reject} & \text{if } \mathbf{M} \text{ accepts } < \mathbf{M} > \end{cases}$$

Undecidable problem

Next consider a new decider **D** that uses **H** as subprocedure:

D = "On input string $< \mathbf{M} >$:

1) Run **H** on input string $< \mathbf{M}, < \mathbf{M} >>$

2) If **H** rejects, accept; **H** accepts, reject"

$$\mathbf{D}(< \mathbf{M} >) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ does not accepts } < \mathbf{M} > \\ \text{reject} & \text{if } \mathbf{M} \text{ accepts } < \mathbf{M} > \end{cases}$$

What if we run **D** on $< \mathbf{D} >$:

$$\mathbf{D}(< \mathbf{D} >) = \begin{cases} \text{accept} & \text{if } \mathbf{D} \text{ does not accepts } < \mathbf{D} > \\ \text{reject} & \text{if } \mathbf{D} \text{ accepts } < \mathbf{D} > \end{cases}$$

But this is impossible.

Undecidable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\mathrm{TM}} = \{< \mathbf{M}, s > | \ \mathbf{M} \text{ accepts } s\}$

Fact: cannot find a TM that decides A_{TM}

Undecidable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\text{TM}} = \{< \mathbf{M}, s >| \ \mathbf{M} \text{ accepts } s\}$

Fact: cannot find a TM that decides A_{TM}

Language A_{TM} is undecidable.

Acceptance problem for TM cannot be solved algorithmically.

Unrecognizable problem

Are there languages not even Turing-recognizable?

Unrecognizable problem

Fact: A language L is decidable iff
L is recognizable and $L^{co} := \Sigma^* - L$ is recognizable

Unrecognizable problem

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L is recognizable and $L^{co} := \Sigma^* - L$ is recognizable

1) If L is decidable, then L^{co} is also decidable.
So L is recognizable and L^{co} is recognizable.

Unrecognizable problem

Fact: A language L is decidable iff
L is recognizable and $L^{co} := \Sigma^* - L$ is recognizable

1) If L is decidable, then L^{co} is also decidable.
So L is recognizable and L^{co} is recognizable.

2) If L is recognizable and L^{co} is recognizable,
then let $\mathbf{M}$ and $\mathbf{M}^{co}$ be the recognizers.
The following is a decider for L:

$\mathbf{H}$ = "On input string s:

1) Run s on $\mathbf{M}$ and $\mathbf{M}^{co}$ in parallel

2) If $\mathbf{M}$ accepts, accept; if $\mathbf{M}^{co}$ accepts, reject"

Unrecognizable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\mathrm{TM}} = \{< \mathbf{M}, s >| \ \mathbf{M} \text{ accepts } s\}$ (undecidable)

Unrecognizable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\mathrm{TM}} = \{< \mathbf{M}, s >\mid \mathbf{M} \text{ accepts } s\}$ (undecidable)

So either A_{TM} is not recognizable, or A_{TM}^{co} is not recognizable.

Unrecognizable problem

Acceptance problem for TM:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ accepts s

Consider $A_{\mathrm{TM}} = \{< \mathbf{M}, s >| \ \mathbf{M} \text{ accepts } s\}$ (undecidable)

So either A_{TM} is not recognizable, or A^{co}_{TM} is not recognizable.

We know A_{TM} is recognizable.
Therefore A^{co}_{TM} is not recognizable.

Recap (language relations)

More undecidable problems

TM halting problem:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ halts on s

More undecidable problems

TM halting problem:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ and a string $s \in \Sigma^*$, test if $\mathbf{M}$ halts on s

Consider $H_{\text{TM}} = \{< \mathbf{M}, s >| \ \mathbf{M} \text{ halts on } s\}$

Claim: H_{TM} is undecidable

More undecidable problems

Proof: Assume H_{TM} is decidable (to obtain a contradiction)

Then there is a decider $\mathbf{H}$ for H_{TM} (always halts):

$$\mathbf{H}(<\mathbf{M}, s>) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ halts on } s \\ \text{reject} & \text{if } \mathbf{M} \text{ loops on } s \end{cases}$$

More undecidable problems

Proof: Assume H_{TM} is decidable (to obtain a contradiction)

Then there is a decider $\mathbf{H}$ for H_{TM} (always halts):

$$\mathbf{H}(<\mathbf{M}, s>) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ halts on } s \\ \text{reject} & \text{if } \mathbf{M} \text{ loops on } s \end{cases}$$

The following is a decider for A_{TM}:

$\mathbf{D} =$ "On input string $<\mathbf{M}, s>$:

1) Run $\mathbf{H}$ on $<\mathbf{M}, s>$
2) If $\mathbf{H}$ rejects, reject (looping is rejected)
3) If $\mathbf{H}$ accepts, run s on $\mathbf{M}$ from the initial state q_0
4) If s ends at q_a, accept; if ends at q_r, reject"

But this is impossible.

More undecidable problems

Strategy to prove that a language is undecidable:

1) Assume the language is decidable,
so there is a decider **H**

2) Construct another decider **D** for
another language known to be undecidable

More undecidable problems

TM emptyness testing problem:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$,
test if $L_a(\mathbf{M}) = \emptyset$

More undecidable problems

TM emptyness testing problem:

Given a TM $\mathbf{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$,
test if $L_a(\mathbf{M}) = \emptyset$

Consider $E_{\text{TM}} = \{< \mathbf{M} > | \ L_a(\mathbf{M}) = \emptyset\}$

Claim: E_{TM} is undecidable

More undecidable problems

Proof: Assume E_{TM} is decidable (to obtain a contradiction)
Then there is a decider $\mathbf{E}$ for E_{TM} (always halts):

$$\mathbf{E}(<\mathbf{M}>) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ accepts no string} \\ \text{reject} & \text{if } \mathbf{M} \text{ accepts some string} \end{cases}$$

More undecidable problems

Proof: Assume E_{TM} is decidable (to obtain a contradiction)
Then there is a decider $\mathbf{E}$ for E_{TM} (always halts):

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Want to construct a decider $\mathbf{D}$ for A_{TM} using $\mathbf{E}$.

$\mathbf{D} =$ "On input string $<\mathbf{M}, s>$:

More undecidable problems

Proof: Assume E_{TM} is decidable (to obtain a contradiction)

Then there is a decider $\mathbf{E}$ for E_{TM} (always halts):

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Want to construct a decider $\mathbf{D}$ for A_{TM} using $\mathbf{E}$.

$\mathbf{D} =$ "On input string $< \mathbf{M}, s >$:

use $\mathbf{M}$ and s to construct another TM $\mathbf{M}'$ s.t.

$\mathbf{M}'$ accepts s iff $\mathbf{M}$ accepts s, but rejects all other strings;

then run decdier $\mathbf{E}$ on $< \mathbf{M}' >$

More undecidable problems

Proof: Assume E_{TM} is decidable (to obtain a contradiction)
Then there is a decider $\mathbf{E}$ for E_{TM} (always halts):

$$\mathbf{E}(<\mathbf{M}>) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ accepts no string} \\ \text{reject} & \text{if } \mathbf{M} \text{ accepts some string} \end{cases}$$

Want to construct a decider $\mathbf{D}$ for A_{TM} using $\mathbf{E}$.

$\mathbf{D} =$ "On input string $< \mathbf{M}, s >$:

1) Construct the following TM $\mathbf{M}'$:

$\mathbf{M}' =$ 'On input string $< w >$:

i) If $w \neq s$, reject.

ii) If $w = s$, run s on $\mathbf{M}$. If $\mathbf{M}$ accepts s, accept'

More undecidable problems

Proof: Assume E_{TM} is decidable (to obtain a contradiction)
Then there is a decider $\mathbf{E}$ for E_{TM} (always halts):

$$\mathbf{E}(< \mathbf{M} >) = \begin{cases} \text{accept} & \text{if } \mathbf{M} \text{ accepts no string} \\ \text{reject} & \text{if } \mathbf{M} \text{ accepts some string} \end{cases}$$

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i) If $w \neq s$, reject.

ii) If $w = s$, run s on $\mathbf{M}$. If $\mathbf{M}$ accepts s, accept'

2) Run $\mathbf{E}$ on $< \mathbf{M}' >$

3) If $\mathbf{E}$ rejects, accept; if $\mathbf{E}$ accepts, reject"

But this is impossible.

Recap

Acceptance/halting problem for TM is undecidable

Emptyness testing problem for TM is undecidable

Fact: testing any properties of the languages recognized by TM is undecidable (Rice's theorem)