Finite Automaton

Alphabet

Alphabet Σ is a finite set of symbols:

$$\Sigma = \{\alpha, \beta, \gamma, \sigma, ...\}$$

Strings

A string on Σ is a finite sequence of symbols from Σ:

$$s = \sigma_1 \sigma_2 \cdots \sigma_k, \quad \sigma_i \in \Sigma, \ i = 1, ..., k$$

Denote by Σ^+ the set of all strings on Σ

Strings

Let ϵ be the empty string ($\epsilon \notin \Sigma^+$)

Write $\Sigma^* = \{\epsilon\} \cup \Sigma^+$

Examples

1) $\Sigma = \{\alpha\}$

$\Sigma^+ = ?$ $\{\alpha, \alpha\alpha, \alpha\alpha\alpha, \cdots\}$ $(= \alpha^k)$

$\Sigma^* = ?$ $\{\varepsilon, \alpha, \alpha\alpha, \alpha\alpha\alpha, \cdots\}$

2) $\Sigma = \{\alpha, \beta\}$

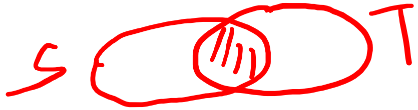
$\Sigma^+ = ?$ $\{\alpha, \beta, \alpha\beta, \alpha\alpha\beta, \beta\beta\alpha, \alpha\alpha, \beta\beta, \cdots\}$

$\Sigma^* = ?$ $\{\varepsilon\} \cup \Sigma^+$

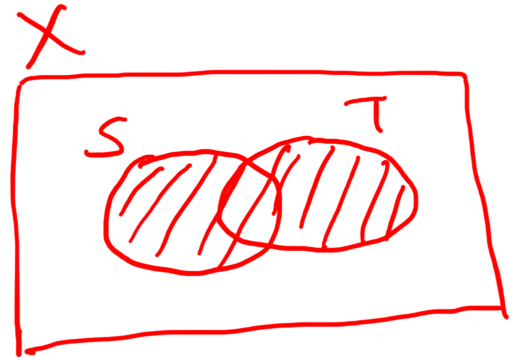
3) $\Sigma = \emptyset$

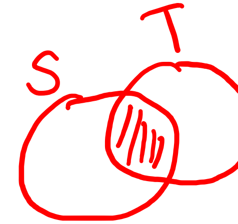
$\Sigma^+ = ?$ ϕ

$\Sigma^* = ?$ $\{\varepsilon\} \cup \phi = \{\varepsilon\}$



Aside: Set operations





Let X be a set and
S, T be two subsets of X

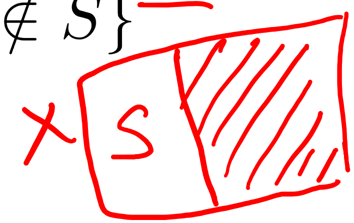
$\in T\}$

The *union* of S and T is $S \cup T = \{x \in X \mid x \in S \text{ or } x \in T\}$

The *intersection* of S and T is $S \cap T = \{x \in X \mid x \in S \cancel{\text{ and } x \in T}\}$

and $x \in T$

The *complement* of S is $S^c = \{x \in X \mid x \notin S\}$



(note: $(S^c)^c = S$)

De Morgan's Law:

$$S^c \cap T^c = (S \cup T)^c$$

$$S^c \cup T^c = (S \cap T)^c$$

Catenation of strings

- Catenation function $cat : \Sigma^* \times \Sigma^* \to \Sigma^*$

$$cat(s_1, s_2) = s_1 s_2, \;\; s_1, s_2 \in \Sigma^*$$

s_1 s_2

$s_1 s_2$

ϵ is the unit element of catenation:

$$cat(s, \epsilon) = s = cat(\epsilon, s), \;\; s \in \Sigma^*$$

$\cancel{s_2 = \beta\alpha}$

$s_1 = \alpha,$
$s_2 = \beta\alpha$

cat is *not* commutative:

$s_2 s_1 = \beta\alpha\alpha$ $\quad$ $s_1 s_2 = \alpha\beta\alpha,$

$$cat(s_1, s_2) \neq cat(s_2, s_1), \;\; s_1, s_2 \in \Sigma^*$$

String length

$s = a\beta a\beta$

$|s| = 4$

Let $s \in \Sigma^*$ be a string.

Length $|s|$:

$$\text{if } s = \sigma_1 \cdots \sigma_k \in \Sigma^+, \text{ then } |s| = k$$

$$\text{if } s = \epsilon, \text{ then } |s| = 0$$

$$|cat(s,t)| = |s| + |t|, \quad s,t \in \Sigma^* \text{ (why?)}$$

State transition function

$$q \xrightarrow{\sigma} q'$$

$$\delta(q,\sigma) = q'$$

It's convenient to extend $\delta : Q \times \Sigma \to Q$
to $\hat{\delta} : Q \times \Sigma^* \to Q$

$$q_1 \xrightarrow{\sigma_1} q_2 \xrightarrow{\sigma_2} q_3 \xrightarrow{\sigma_3} q_4$$

$$\delta(q_1, \sigma_1) = q_2$$

$$\delta(q_2, \sigma_2) = q_3$$

$$\delta(q_3, \sigma_3) = q_4$$

$$\boxed{\hat{\delta}(q_1, \sigma_1\sigma_2\sigma_3) = q_4}$$

$$\hat{\delta}(q,\epsilon) = q, \quad q \in Q$$

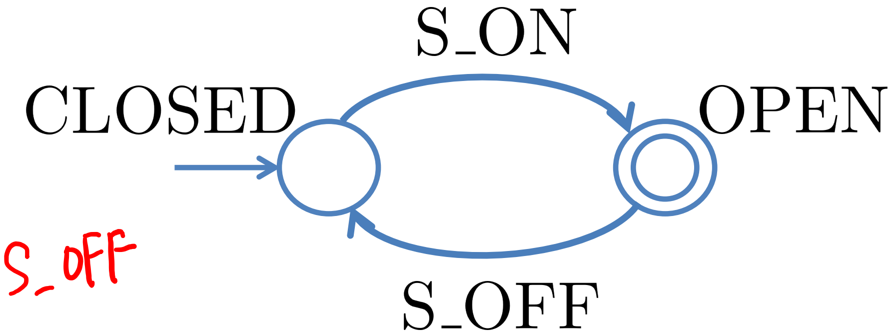
$$\hat{\delta}(q,\sigma) = \delta(q,\sigma), \quad q \in Q, \sigma \in \Sigma$$

$$\hat{\delta}(q,\underline{s\sigma}) = \delta(\underline{\hat{\delta}(q,s)},\sigma), \quad q \in Q, s \in \Sigma^*, \sigma \in \Sigma$$

$$q \xrightarrow{s} \bullet \xrightarrow{\sigma} \bullet$$

Henceforth drop ^ and just write δ.

Example: automatic door



s = S_ON . S_OFF

δ(CLOSED, s) = CLOSED

δ(OPEN, ε) = OPEN

Languages

- Language L on Σ is any subset of Σ^*, i.e.

$$L \subseteq \Sigma^*$$

(A language on Σ is a set of strings on Σ)

Examples of languages

1) $\emptyset$ $\subseteq \Sigma^*$ (the empty language)

2) $\{\epsilon\}$ $\subseteq \Sigma^*$ (the language containing only ϵ)

3) Σ^+ $\subseteq \Sigma^*$ (the language containing every string but ϵ)

but ε)

4) Σ^* $\subseteq \Sigma^*$ (the language containing all strings ~~on Σ~~)

on Σ)

Examples of languages

Let $\Sigma = \{\alpha, \beta\}$

1) $L_1 = \{\alpha^n | n = 0, 1, 2, ...\} = \alpha^*$

$= \{\varepsilon, \alpha, \alpha\alpha, \alpha\alpha\alpha, \cdots\} \subseteq \Sigma^*$

2) $L_2 = \{\alpha^n \beta^n | n = 0, 1, 2, ...\}$

$= \{\varepsilon, \alpha\beta, \alpha\alpha\beta\beta, \alpha\alpha\alpha\beta\beta\beta, \ldots\}$

3) $L_3 = \{(\alpha\beta)^n | n = 0, 1, 2, ...\}$

$= \{\varepsilon, \alpha\beta, \alpha\beta\alpha\beta, \alpha\beta\alpha\beta\alpha\beta, \cdots\}$

4) $L_4 = \{s | \#\alpha(s) = \#\beta(s)\}$, $\alpha\beta, \beta\alpha$

where $\#\alpha(s)$: the number of α's in s

$\#\beta(s)$: the number of β's in s

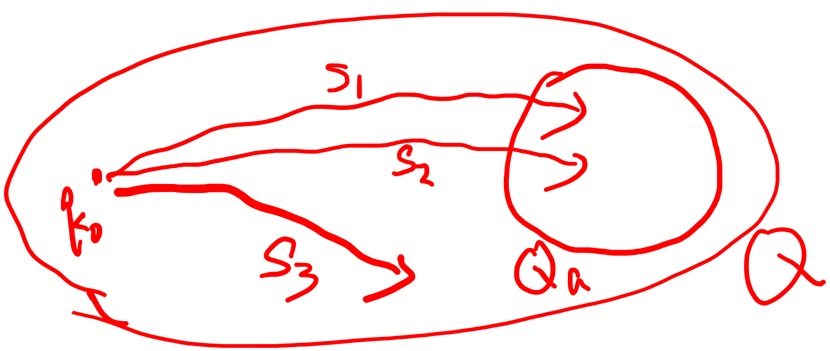
Accepted/recognized languages

Let $\mathbf{G} = (Q, \Sigma, \delta, q_0, Q_m)$ be an automaton.

The accepted/recognized language of $\mathbf{G}$ is:

$$L_a(\mathbf{G}) := \{s \in \Sigma^* \mid \delta(q_0, s) \in Q_a\}$$

"Set of all strings that end in Q_a"

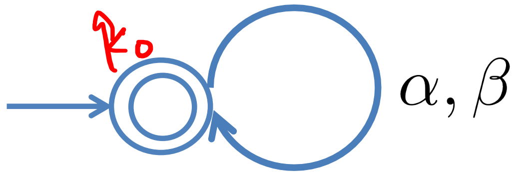


Examples

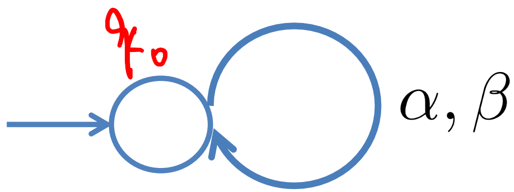
$\delta(q_0, \alpha) = q_0$
$\delta(q_0, \beta) = q_0$

$Q = \{q_0\}$
$\Sigma = \{\alpha, \beta\}$
$Q_a = \{q_0\}$

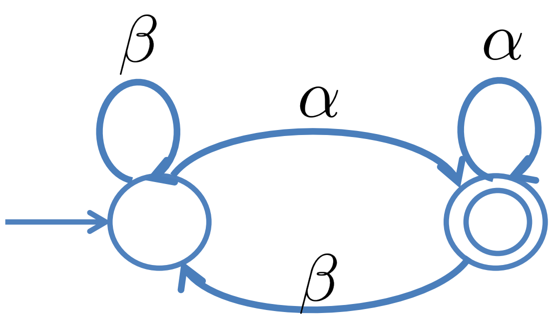
$\mathbf{G} = (Q, \Sigma, \delta, q_0, Q_a)$



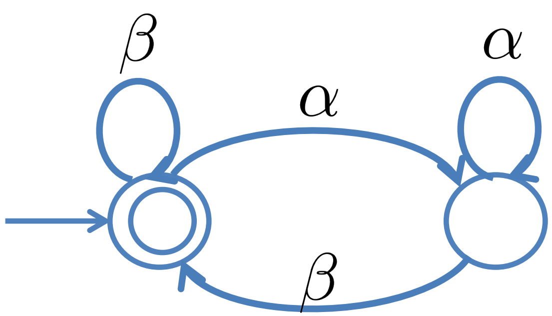
$L_a(G) = \{\varepsilon, \alpha, \beta, \alpha\alpha, \cdots\} = \Sigma^*$
$= \Sigma^*$



$L_a(G) = \emptyset$



$L_a(G) = \{\alpha, \alpha\alpha, \alpha\beta\alpha, \cdots\}$



$L_a(G) = \{\varepsilon, \beta, \beta\beta, \beta\alpha\beta, \cdots$

Automaton design

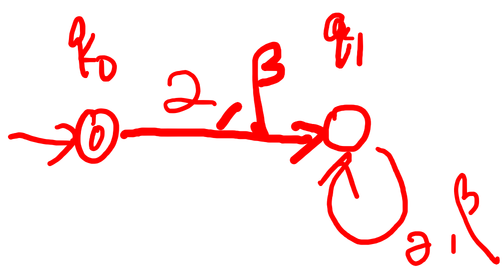
Reverse problem: Given a language $L \subseteq \Sigma^*$ design a finite automaton $\mathbf{G}$ to recognize L

$\Sigma = \{\alpha, \beta\}$

1) Σ^*

$L_a(G) = \Sigma^*$

2) $\{\epsilon\}$

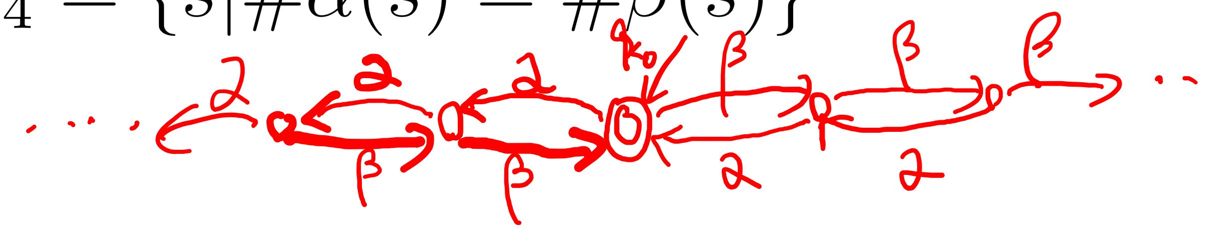


Automaton design

Reverse problem: Given a language $L \subseteq \Sigma^*$ design a finite automaton $\mathbf{G}$ to recognize L

3) $L_3 = \{(\alpha\beta)^n | n = 0, 1, 2, \ldots\}$: homework 1

4) $L_4 = \{s | \#\alpha(s) = \#\beta(s)\}$



Regular languages

It is not always possible to design
a finite automaton $\mathbf{G}$ to recognize a given L

Defn. A language $L \subseteq \Sigma^*$ is a *regular language*
if L can be recognized by a finite automaton
i.e. there exists $\mathbf{G}$ s.t. $L_a(\mathbf{G}) = L$