Example

Consider CFG $G = (\mathcal{V}, \Sigma, \mathcal{R}, S)$ where
$\mathcal{V} = \{V\}$
$\Sigma = \{(,)\}$
$\mathcal{R} = \{V \longrightarrow (V) \mid VV \mid \epsilon\ \}$

$S = V$

Give a derivation for $S \longrightarrow^{*} ((()))$

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Example

Consider CFG $G = (\mathcal{V}, \Sigma, \mathcal{R}, S)$ where

$\mathcal{V} = \{V_1, V_2, V_3\}$

$\Sigma = \{a, +, \times, (,)\}$

$$\begin{aligned}\mathcal{R} = \{V_1 &\longrightarrow V_1 + V_2 \mid V_2 \\ V_2 &\longrightarrow V_2 \times V_3 \mid V_3 \\ V_3 &\longrightarrow (V_1) \mid a\}\end{aligned}$$

$S = V_1$

Give a derivation for $S \longrightarrow^* a + a \times a$

$$\begin{aligned}V_1 \rightarrow V_1 + V_2 &\rightarrow V_1 + V_2 \times V_3 \\ &\rightarrow V_2 + V_2 \times V_3 \\ &\rightarrow V_3 + V_3 \times V_3 \\ &\rightarrow a + a \times a\end{aligned}$$

Example

Consider CFG $G = (\mathcal{V}, \Sigma, \mathcal{R}, S)$ where

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$S = V_1$

Give a derivation for $S \longrightarrow^* (a + a) \times a$

$$\begin{aligned} V_1 \rightarrow V_2 &\rightarrow V_2 \times V_3 \\ &\rightarrow V_2 \times a \\ &\rightarrow V_3 \times a \rightarrow (V_1) \times a \rightarrow (V_1 + V_2) \times a \\ &\rightarrow (V_2 + V_2) \times a \rightarrow (V_3 + V_3) \times a \rightarrow (a + a) \times a \end{aligned}$$

Context-free languages

Defn. Let $L \subseteq \Sigma^*$.
L is a *context-free language* (CFL) if
there is a context-free grammar G s.t. $L(G) = L$

So $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL
(we know this language is not regular)

G : $A \to 0A1$
$A \to B$
$B \to \varepsilon$

G' : $A \to 0A1$
$A \to \varepsilon$

Context-free languages

Defn. Let $L \subseteq \Sigma^*$.
L is a *context-free language* (CFL) if
there is a context-free grammar G s.t. $L(G) = L$

How about $L = \{1^n 0^n \mid n \geq 0\}$?

$G: \quad A \to 1A0 \mid \varepsilon$

$L(G) = L$

How about $L = \{0^n 1^n \mid n \geq 0\} \cup \{1^n 0^n \mid n \geq 0\}$?

$G: \quad A_0 \to A_1 \mid A_2$

$A_1 \to 0A_1 1 \mid \varepsilon$

$A_2 \to 1A_2 0 \mid \varepsilon$

$L(G) = L$

$L = \{0^n 1^n \mid n \geq 0\}$

Context-free languages

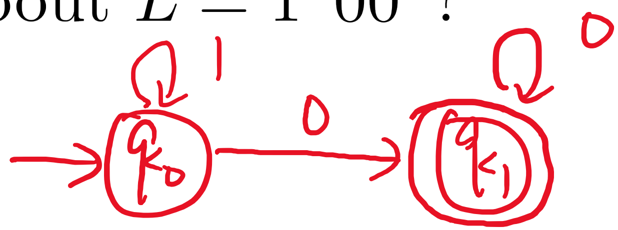
$L = 0^* 1^*$

Defn. Let $L \subseteq \Sigma^*$.
L is a *context-free language* (CFL) if
there is a context-free grammar G s.t. $L(G) = L$

How about $L = 1^*00^*$?

$(q_0, 1, q_0)$
$(q_0, 0, q_1)$
$(q_1, 0, q_1)$

NFA:



$\downarrow$

CFG: $G = (V, \Sigma, R, S)$

$V = \{V_0, V_1\}$, $\Sigma = \{0, 1, \varepsilon\}$

$R = \{V_0 \to 1V_0,\ V_0 \to 0V_1,\ V_1 \to 0V_1,\ V_1 \to \varepsilon\}$

$S = V_0$

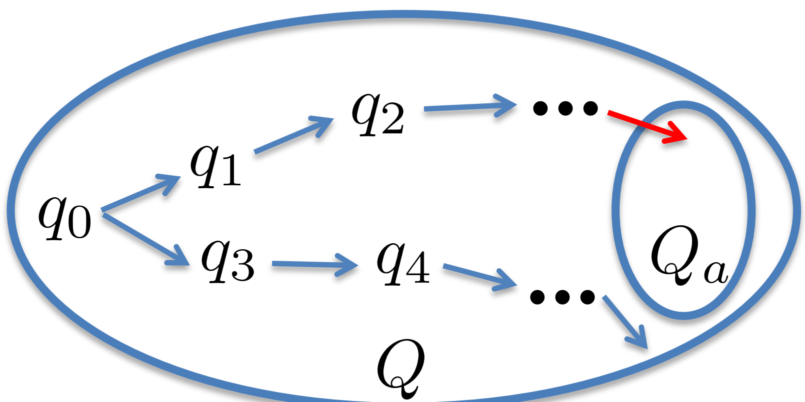
Push-Down Automata

Push-down automata

Regular languages $\Leftrightarrow$ finite automata (NFA, DFA)

Context-free languages $\Leftrightarrow$? Push-down automata (PDA)

finite automata (NFA, DFA)

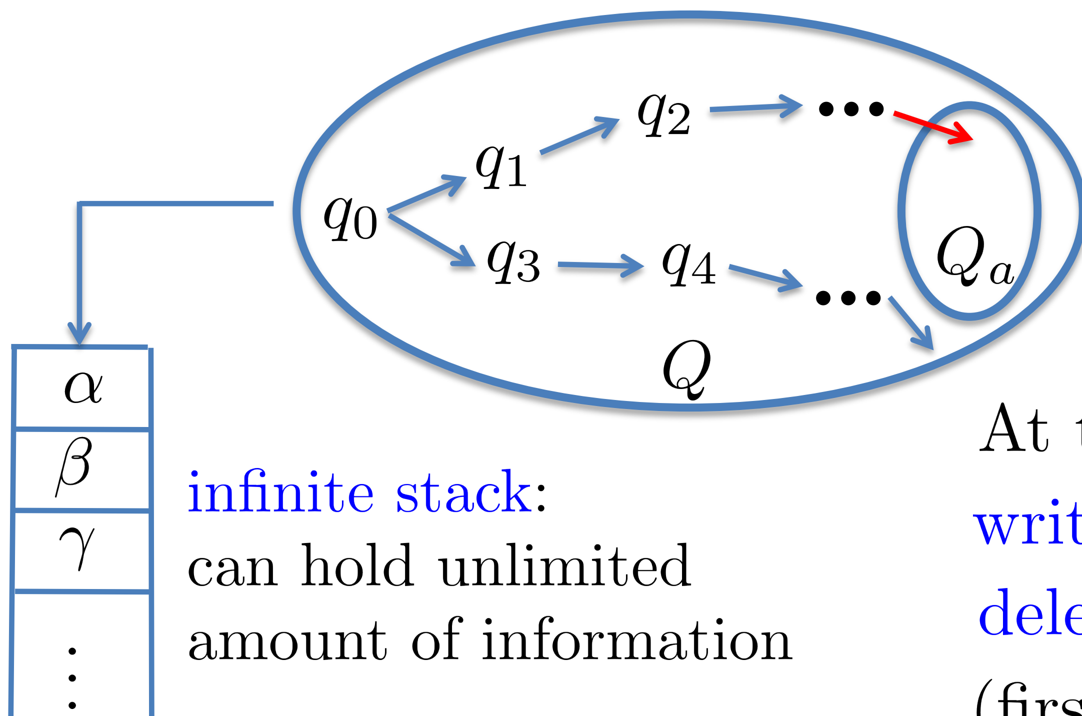


Push-down automata

Regular languages $\Leftrightarrow$ finite automata (NFA, DFA)

Context-free languages $\Leftrightarrow$? Push-down automata (PDA)

push-down automata



infinite stack:
can hold unlimited
amount of information

At top of stack:
write a symbol: push
delete a symbol: pop
(first-in last-out)

Push-down automata

Since $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL,
design a PDA to recognize L.

Initially push a special symbol (say) \$ to the stack

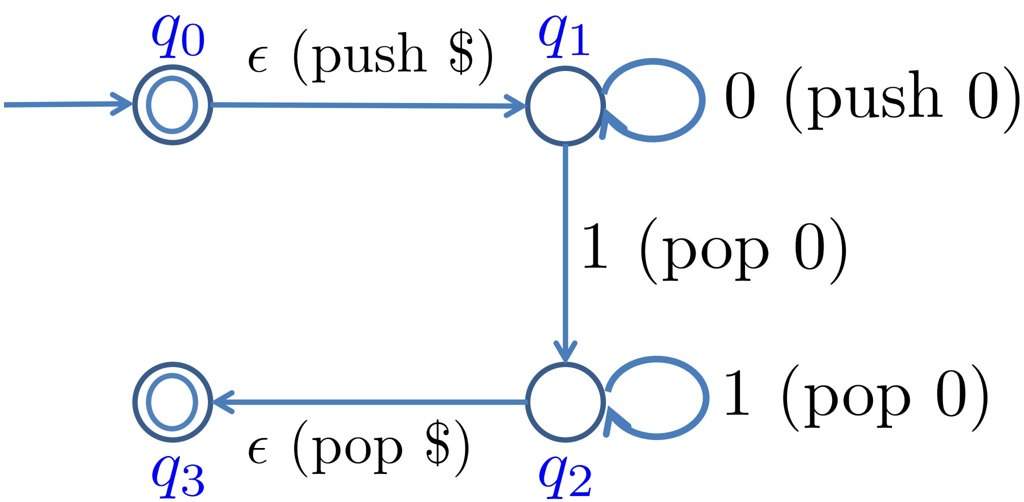
Read an input string s.
Whenever 0 is read, push 0 to the stack.
Whenever 1 is read, pop 0 (if any) from the stack.

If reading s is finished exactly when \$ is at top of stack,
then accept s. (Otherwise reject s.)

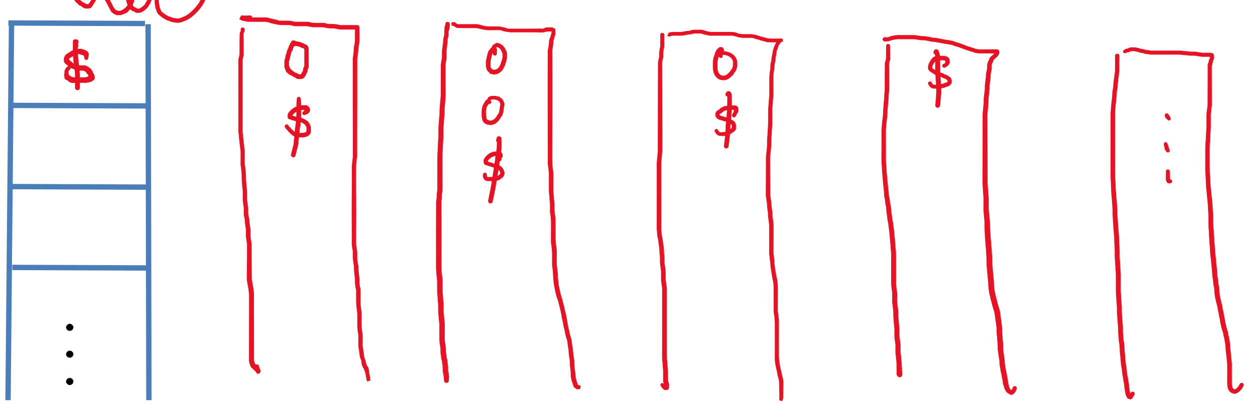
Finally pop \$ from the stack to signal bottom of the stack

Push-down automata

Since $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL,
design a PDA to recognize L.

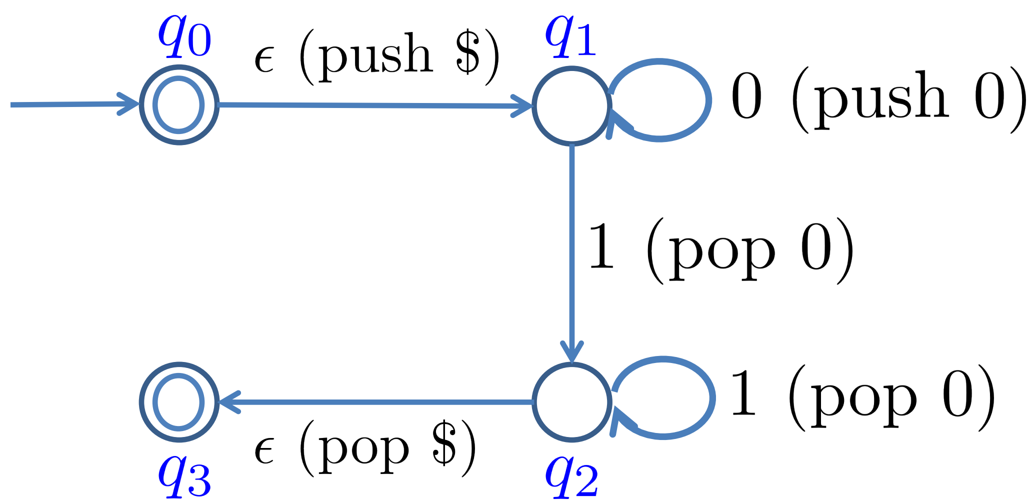


$s = 0011$

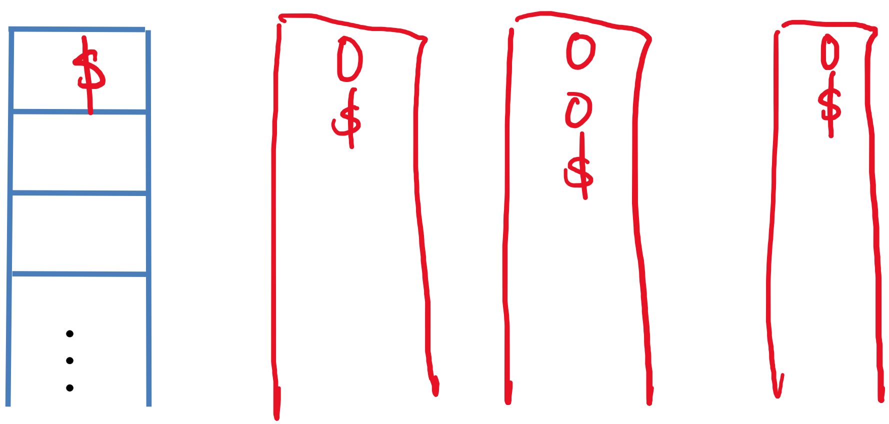


Push-down automata

Since $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL,
design a PDA to recognize L.

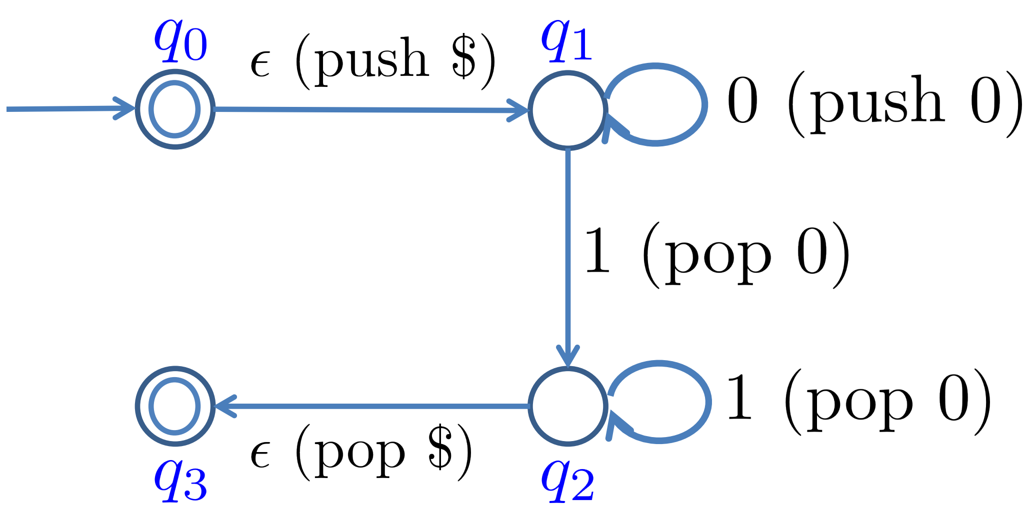


$s = 001$



Push-down automata

Since $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL,
design a PDA to recognize L.



$s = 00111$

Push-down automata

A (nondeterministic) push-down automaton (PDA) $\mathbf{G}$ is a 6-tuple $\mathbf{G} = (Q, \Sigma, \Gamma, \delta, q_0, Q_a)$, where

silent transition

Q : state set; a finite set of states

pop nothing

push nothing

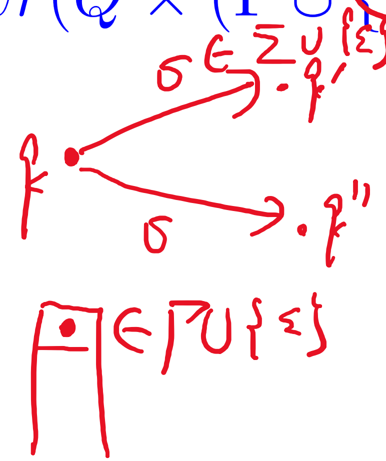
Σ : alphabet; a finite set of symbols

Γ: stack alphabet; a finite set of symbols used in the stack

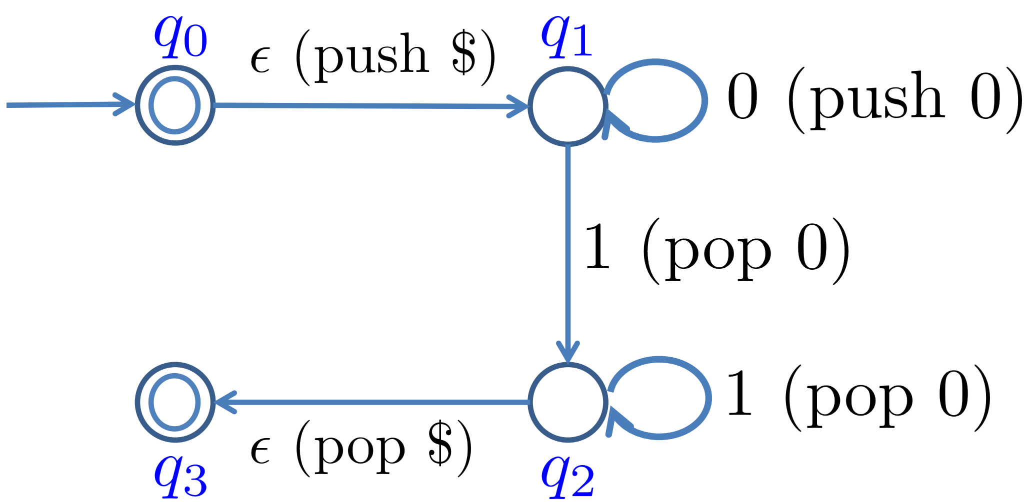
$\delta : Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \rightarrow Pwr(Q \times (\Gamma \cup \{\epsilon\}))$: transition function

$q_0 \in Q$: initial state

$Q_a \subseteq Q$: subset of accept states



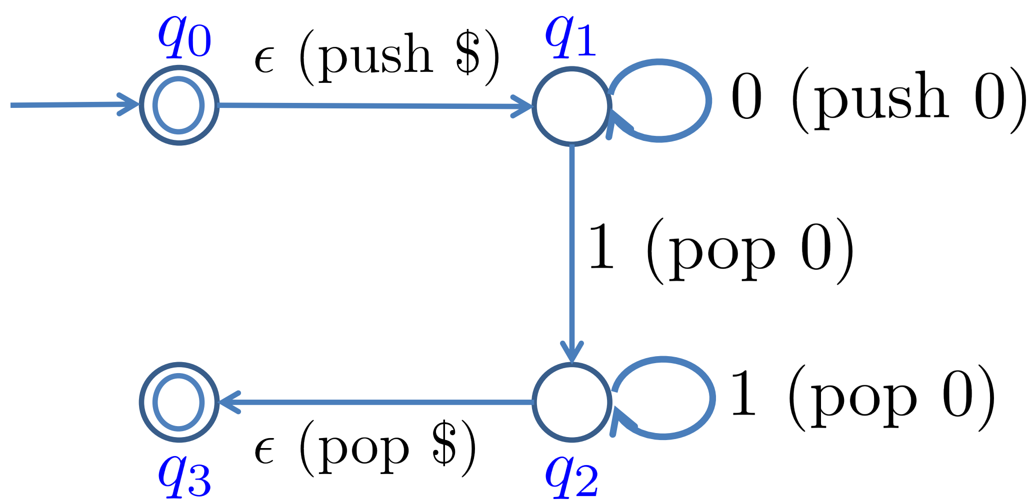
Push-down automata



$Q = \{q_0, q_1, q_2, q_3\}$
$\Sigma = \{0, 1\}$
$\Gamma = \{\$, 0\}$
$q_0 = q_0$
$Q_a = \{q_0, q_3\}$

$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \to Pwr(Q \times (\Gamma \cup \{\varepsilon\}))$

Push-down automata



$$\delta(q_0, \epsilon, \underline{\epsilon}) = \{(q_1, \$)\}$$

$$\delta(q_1, 0, \epsilon) = \{(q_1, 0)\}$$
$$\delta(q_1, 1, 0) = \{(q_2, \varepsilon)\}$$

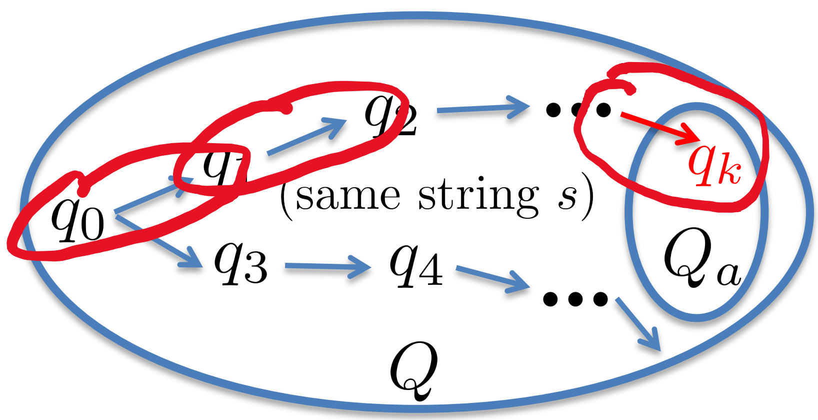
$$\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$$
$$\delta(q_2, \epsilon, \$) = \{(q_3, \varepsilon)\}$$

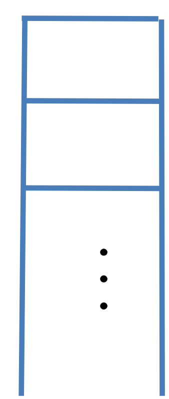
Accepted/recognized strings

Let $\mathbf{G} = (Q, \Sigma, \Gamma, \delta, q_0, Q_a)$ be a PDA.

A string $s = \sigma_1 \cdots \sigma_k (\sigma_i \in \Sigma \cup \{\epsilon\})$ is accepted/recognized by $\mathbf{G}$ if $(\exists q_1, \ldots, q_k \in Q)$

$(q_1, \cdot) \in \delta(q_0, \sigma_1, \cdot), \ldots, (q_k, \cdot) \in \delta(q_{k-1}, \sigma_k, \cdot)$ & $\boxed{q_k \in Q_a}$



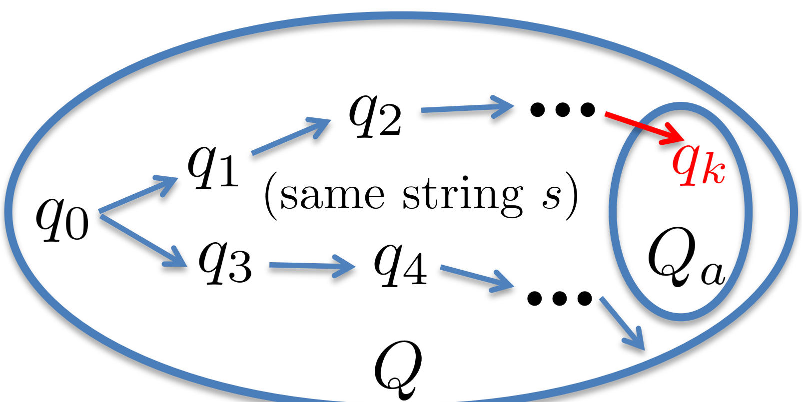


Accepted/recognized languages

Let $\mathbf{G} = (Q, \Sigma, \Gamma, \delta, q_0, Q_a)$ be a PDA.

The language accepted/recognized by $\mathbf{G}$ is:

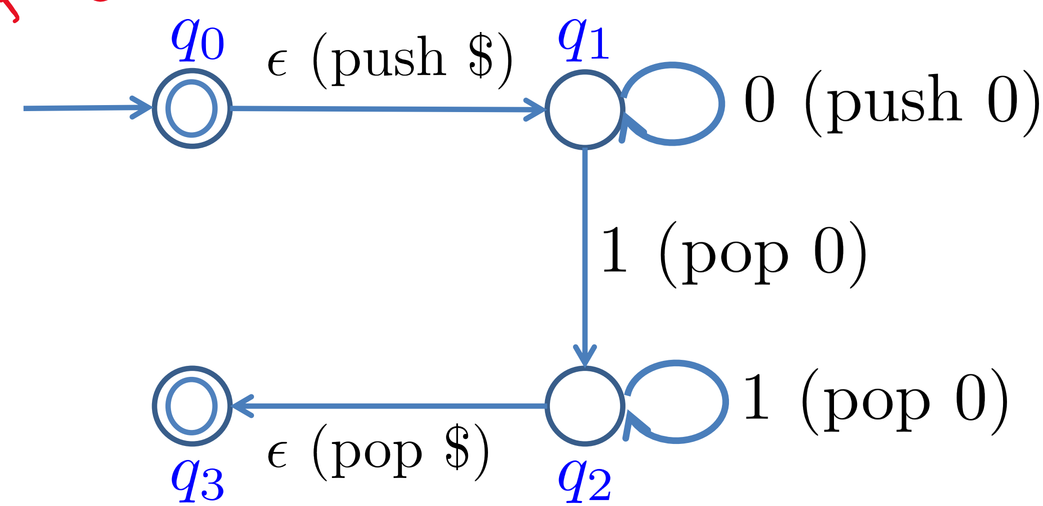
$$L_a(\mathbf{G}) = \{s \in (\Sigma \cup \{\epsilon\})^* \mid s \text{ accepted by } \mathbf{G}\}$$



$L_a(G) = \{0^n 1^n \mid n = 0, 1, 2, \dots\}$

Example

PDA G:



$s = 0011$

$(q_1, \$) \in \delta(q_0, \boxed{\varepsilon}, \varepsilon)$, $(q_1, 0) \in \delta(q_1, \boxed{0}, \varepsilon)$, $(q_1, 0) \in \delta(q_1, \boxed{0}, \varepsilon)$,

$(q_2, \varepsilon) \in \delta(q_1, \boxed{1}, 0)$, $(q_2, \varepsilon) \in \delta(q_2, \boxed{1}, 0)$, $(q_3, \varepsilon) \in \delta(q_2, \boxed{\varepsilon}, \$)$

$\varepsilon 0 0 1 1 \varepsilon = s$