Linear Feedback Control

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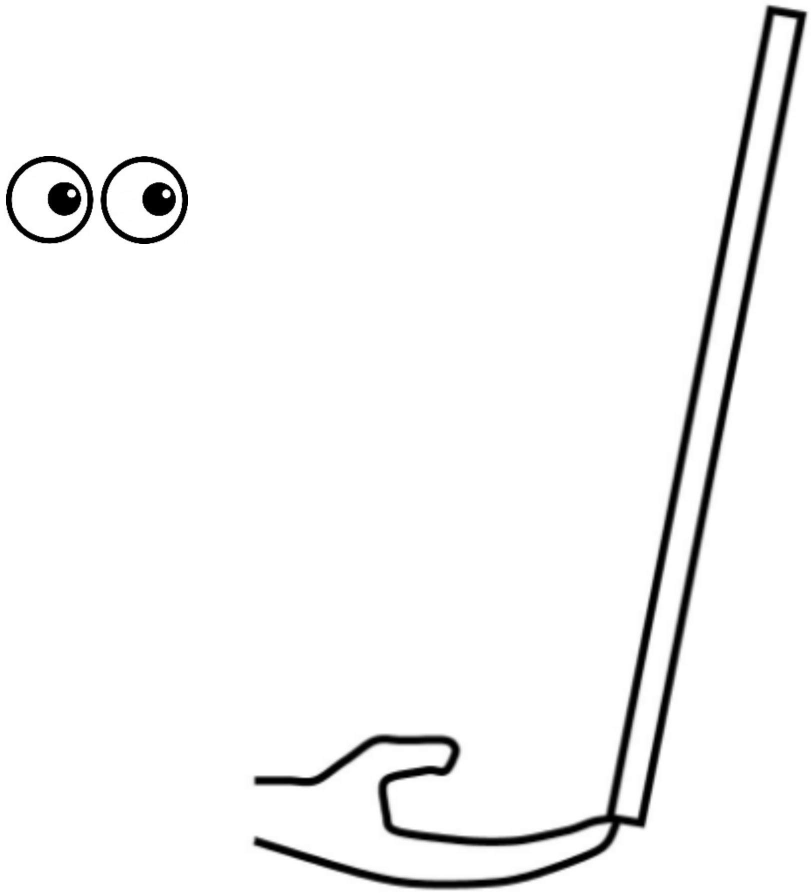
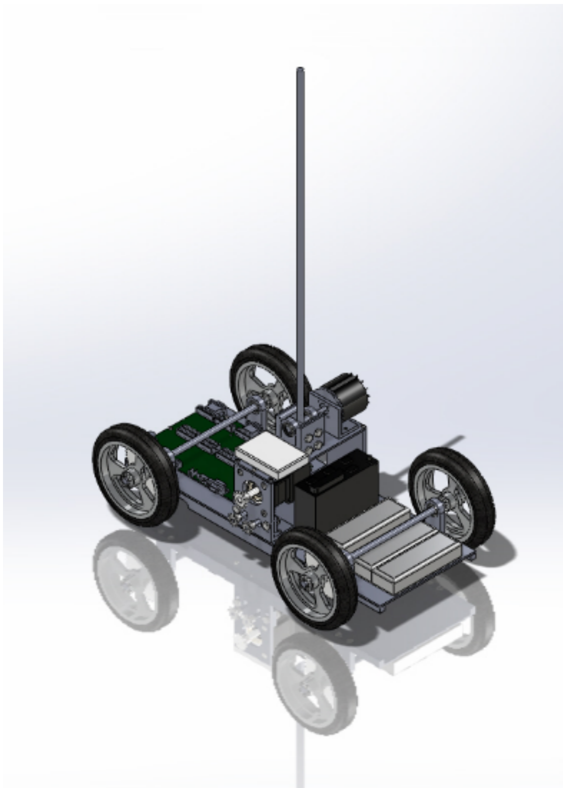
Introduction

What is control?

- System: some entity that dynamically changes over time
- Control: influence the change in a desired way (by observing the system and making decisions)

Balance a stick



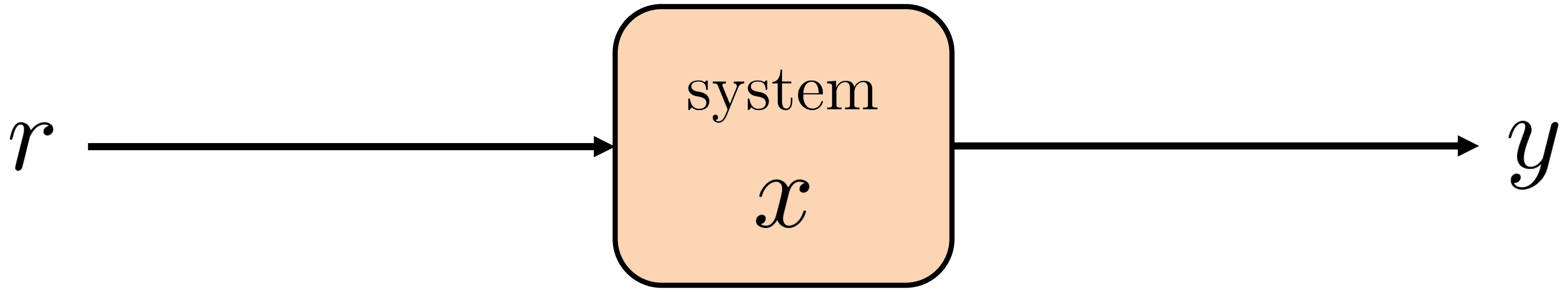


Building blocks

State x : represents what the system is currently doing

Output y : measurement of some aspects of the system

Reference r : what the system is expected to do



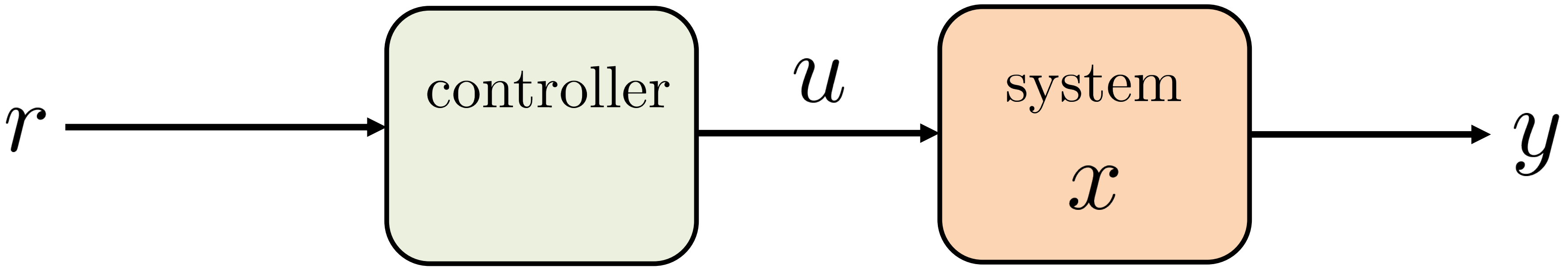
Building blocks

State x : represents what the system is currently doing

Output y : measurement of some aspects of the system

Reference r : what the system is expected to do

Input u : control decisions/signals



Building blocks

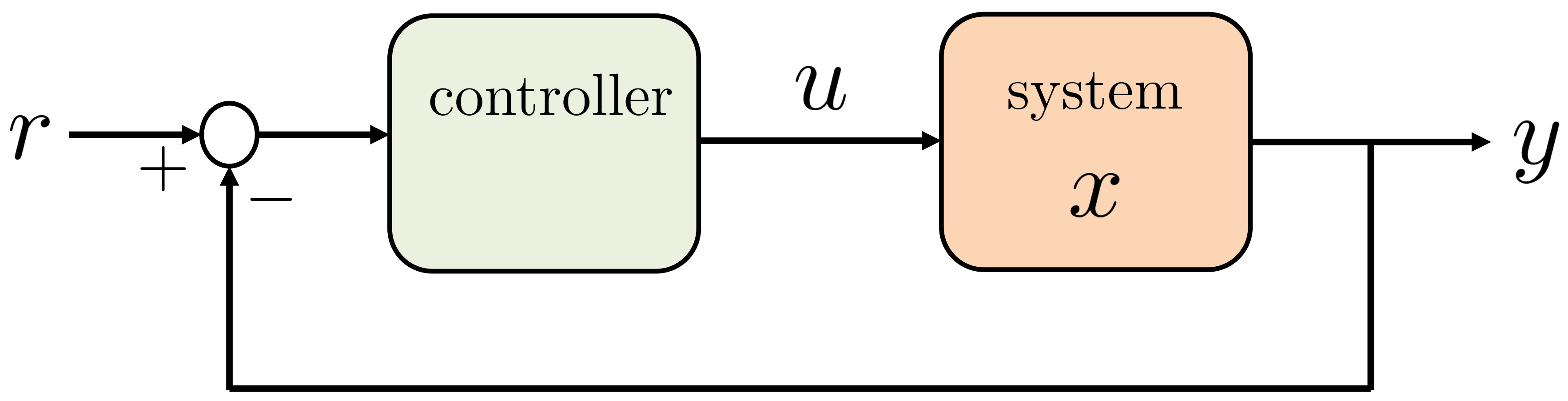
State x : represents what the system is currently doing

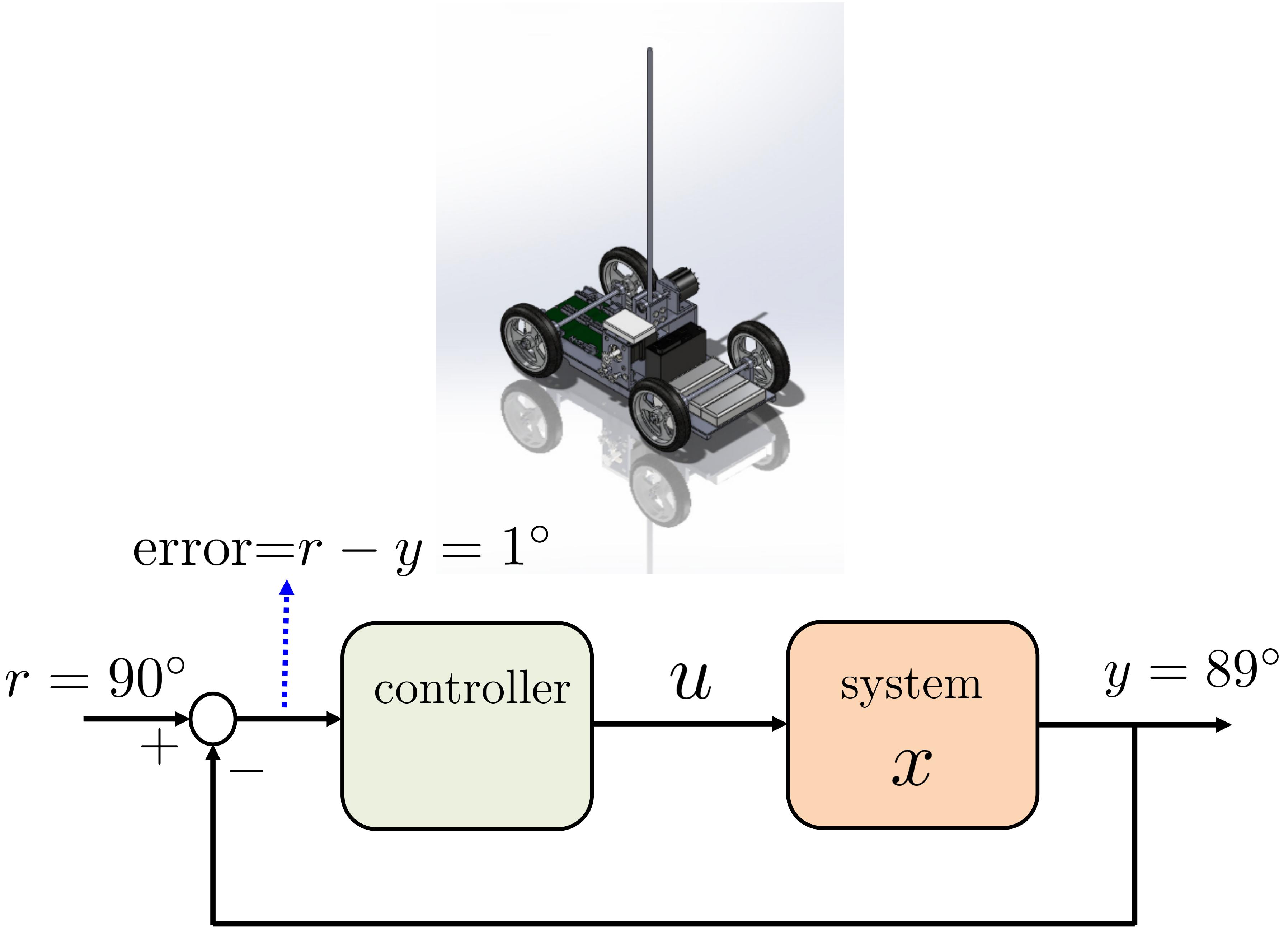
Output y : measurement of some aspects of the system

Reference r : what the system is expected to do

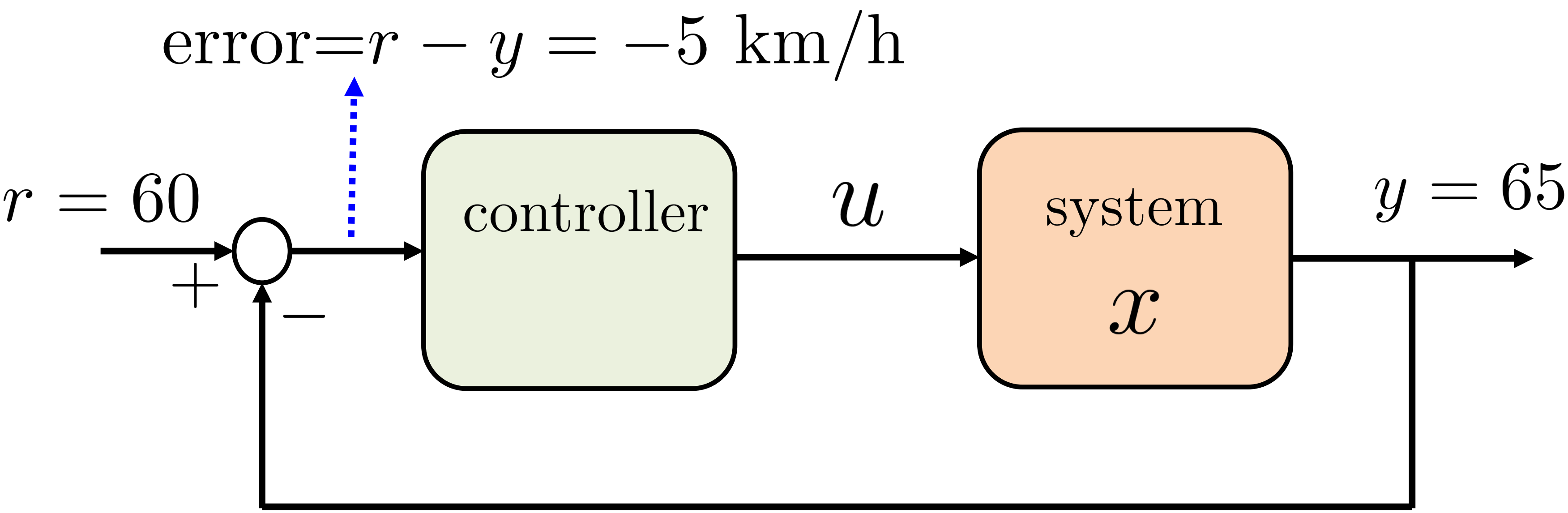
Input u : control decisions/signals

Feedback : mapping from output to input




error=$r - y = 1^\circ$
$r = 90^\circ$
+
−
controller
u
system
x
$y = 89^\circ$

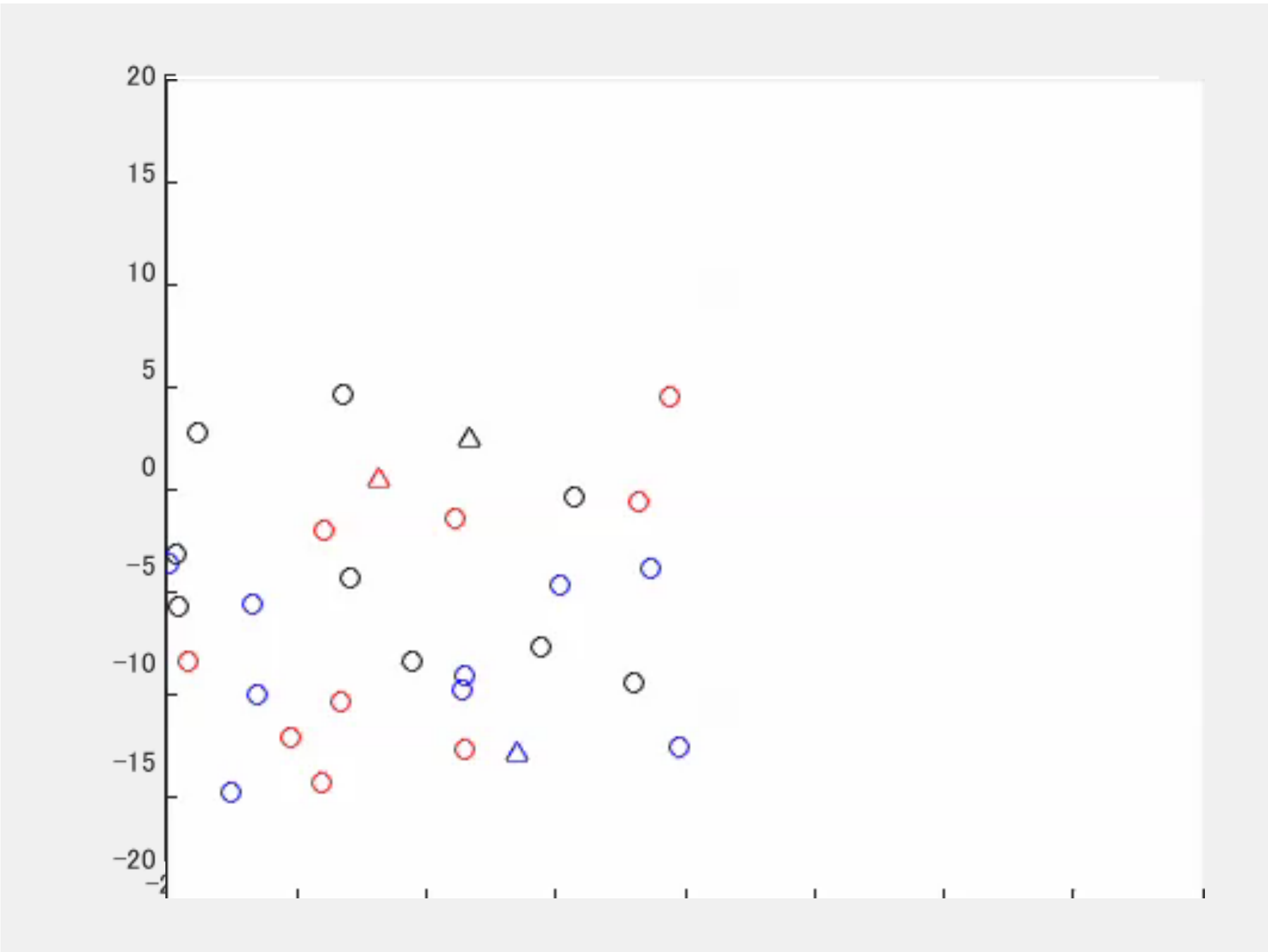

x 1000
KM/H
wiki How to Use Cruise Control on a Ca


error= $r - y = -5$ km/h
$r = 60$
+
−
controller
u
system
x
$y = 65$

Examples

- Cruise control / lane following
- Air-conditioning / thermostat
- Healthcare
- Power/energy network
- Multi-robot formation
- Epidemics
- Stock market

OCU



Group 1 (O)

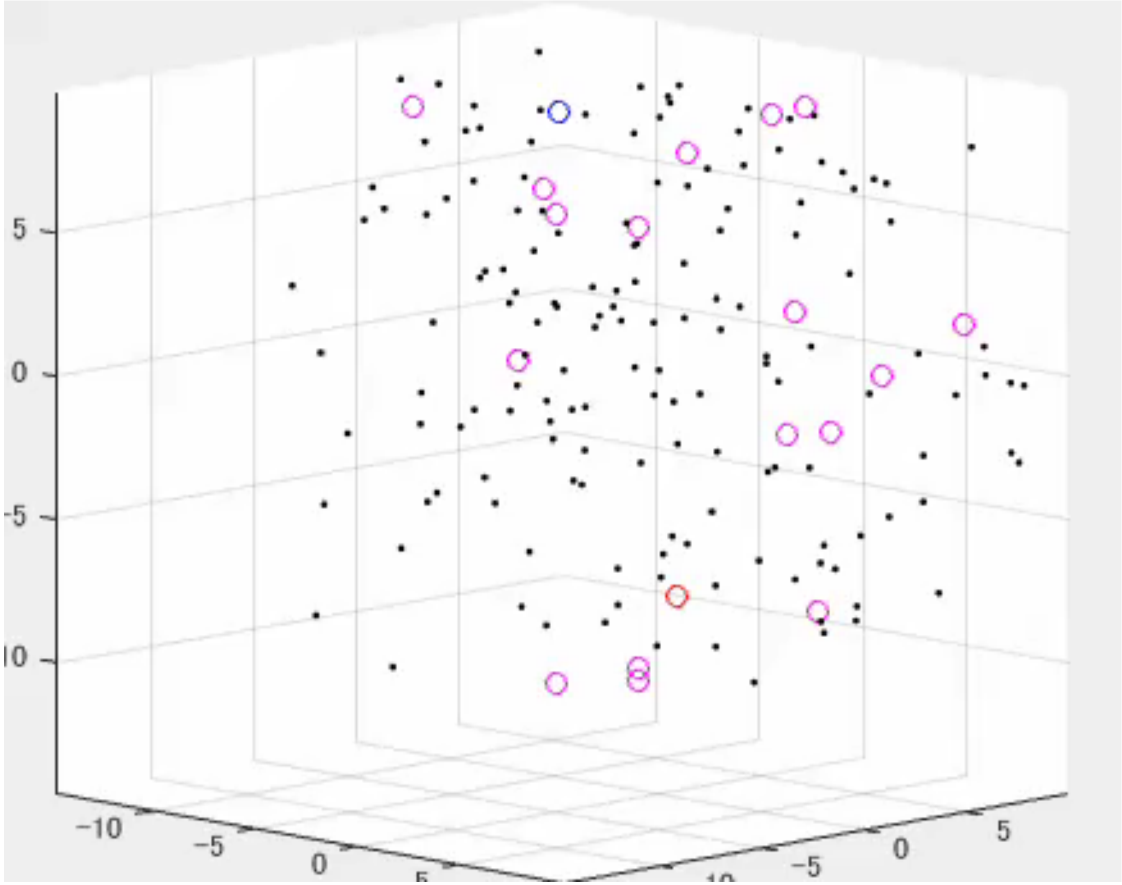
Group 2 (C)

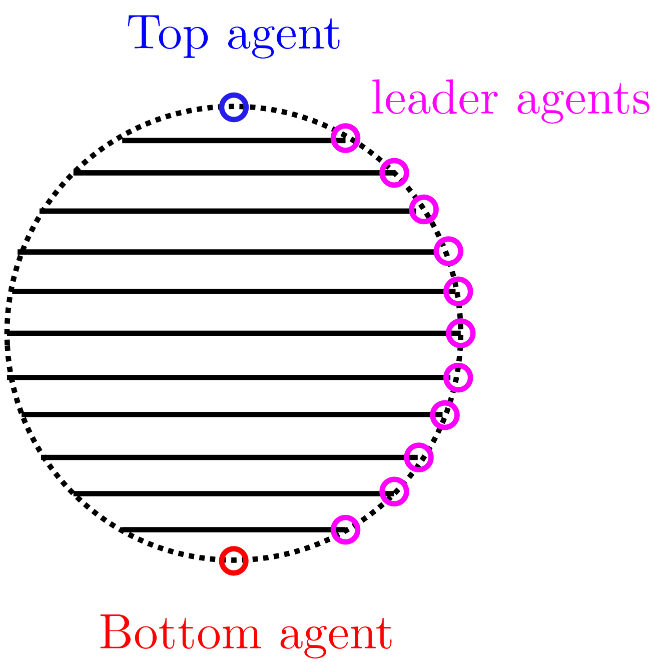
Group 3 (U)

△ : represents a leader

○ : represents a follower

3D Formation





In this course you will learn

1. State models

2. Stability

3. Controllability, state-feedback control design

4. Observability, observer-based control design

5. Tracking and regulation, internal-model control design

6. Optimal control / multi-robot formation control

Website

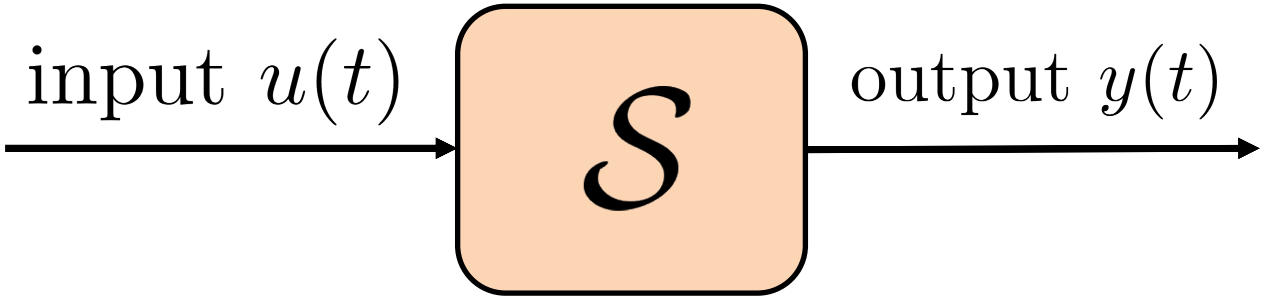
https://www.control.eng.osaka-cu.ac.jp/teaching/linear2019

Cellphone

State Models

What is a "system"?

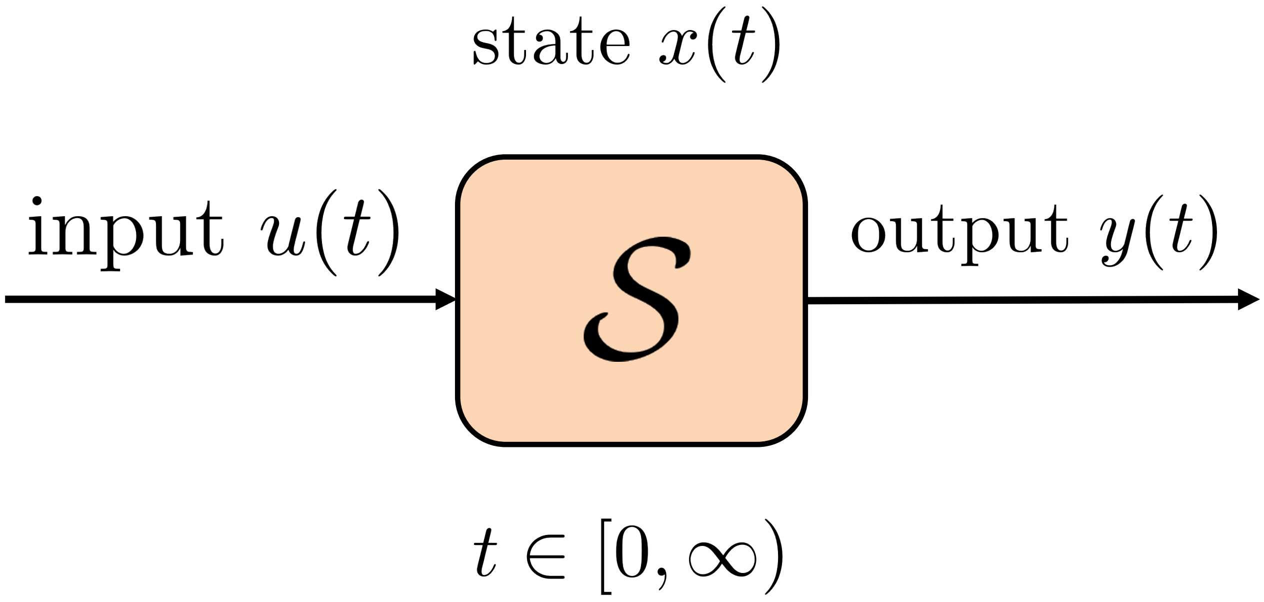
- System: a device that transforms an input signal to an output signal in some specific way



$t \in [0, \infty)$

What is a "system"?

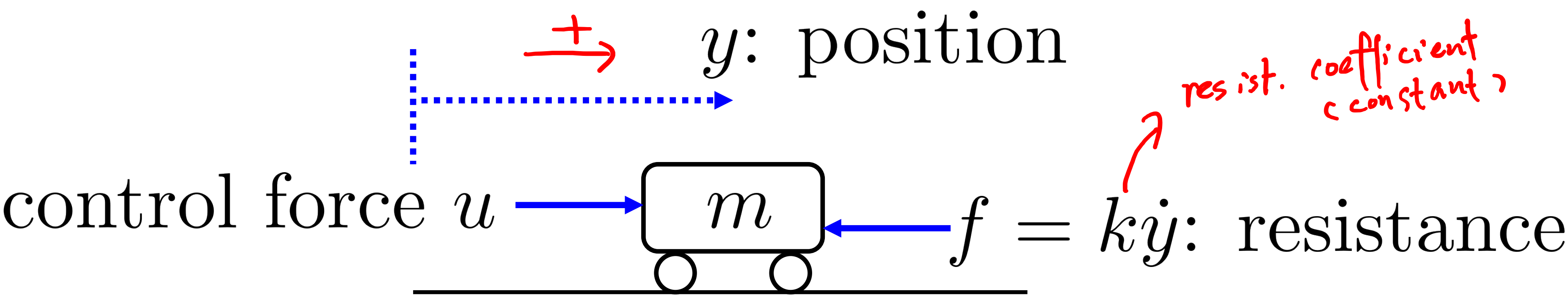
- System: internal state changes over time
- State model: captures such changes



$\dot{y} = \frac{dy}{dt}$ $\quad \ddot{y} = \frac{d^2y}{dt^2}$

Example

- Consider a self-driving car:



(F = ma)

By Newton's second law: $u - f = m\ddot{y}$

$\Rightarrow u - k\dot{y} = m\ddot{y}$

This is a constant-coefficient ODE.

(Ordinary Differential Equation)

Example

ODE: $u - k\dot{y} = m\ddot{y}$

Choose state variables: $x_1 = y$ (pos.), $x_2 = \dot{y}$ (vel.)

Obtain derivatives of state variables:

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = \frac{u - k\dot{y}}{m} = -\frac{k}{m}x_2 + \frac{1}{m}u$$

Put above in matrix form:

$n=2$, $m=1$, $p=1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

Suppose we can measure only position y:

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State model

$$\dot{x} = Ax + Bu$$ (state change)

$$y = Cx$$ (Output/measurement)

n×n n×m m×1 p×n

where $x \in \mathbb{R}^n$: state vector

$u \in \mathbb{R}^m$: (control) input vector ($m \le n$)

$y \in \mathbb{R}^p$: (measurement) output vector ($p \le n$)

$A \in \mathbb{R}^{n \times n}$: state matrix

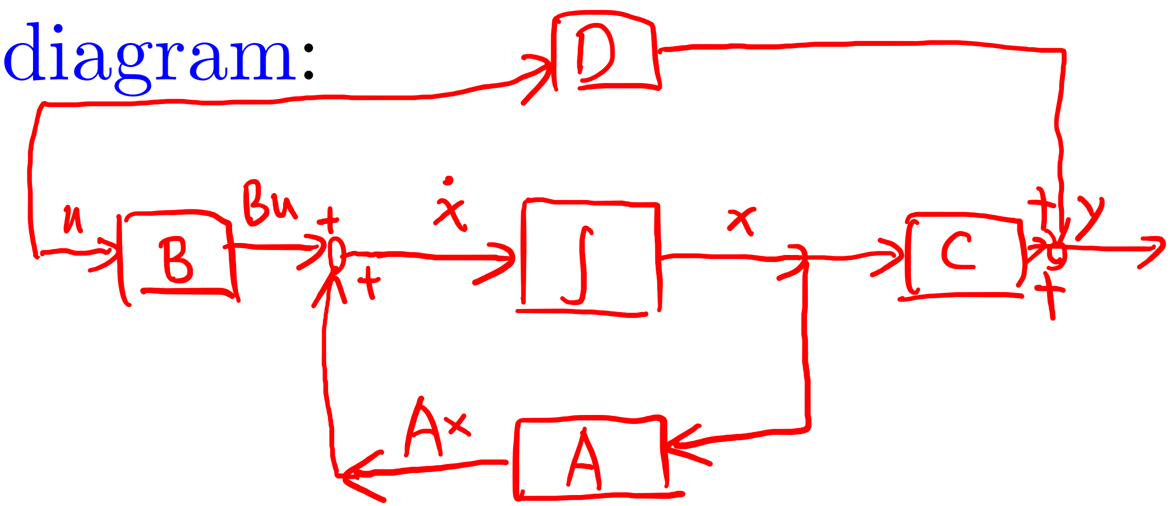
$B \in \mathbb{R}^{n \times m}$: input matrix

$C \in \mathbb{R}^{p \times n}$: output matrix

State model

$$\dot{x} = Ax + Bu$$
$$y = Cx \quad + Du$$

Block diagram:



State model (variation)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$D \in \mathbb{R}^{p \times m}$: input-output matrix

Block diagram:

Convert ODE to state model

Car ODE: $\ddot{y} + \frac{k}{m}\dot{y} = \frac{1}{m}u$

State vector: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$ $\Rightarrow$ $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

More general ODE:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_0 u$$

$$\text{State vector: } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

Convert ODE to state model

Derivative of state vector:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ b_0 u - a_0 y - a_1 \dot{y} - \cdots - a_{n-1} y^{(n-1)} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}}_{A,\ n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}}_{B,\ n\times 1} u$$

Output vector: $y = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$f(t) \Leftrightarrow F(s)$ (time / frequency) $\Rightarrow \dot{f}(t) \Leftrightarrow sF(s)$ ~~$+f(0)$~~, $\ddot{f}(t) \Leftrightarrow s^2F(s)$, ...

Relation of state model with transfer function

More general ODE:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_0 u$$

Take Laplace transform of both sides:

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_1 sY(s) + a_0 Y(s) = b_0 U(s)$$

Transfer function: $G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

State model: $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

Convert ODE to state model

Even more general ODE:

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=$$

$$b_{n-1}u^{(n-1)}+\cdots+b_1\dot{u}+b_0u$$

Take Laplace transform of both sides:

$$s^nY(s)+a_{n-1}s^{n-1}Y(s)+\cdots+a_1sY(s)+a_0Y(s)=$$

$$b_{n-1}s^{n-1}U(s)+\cdots+b_1sU(s)+b_0U(s)$$

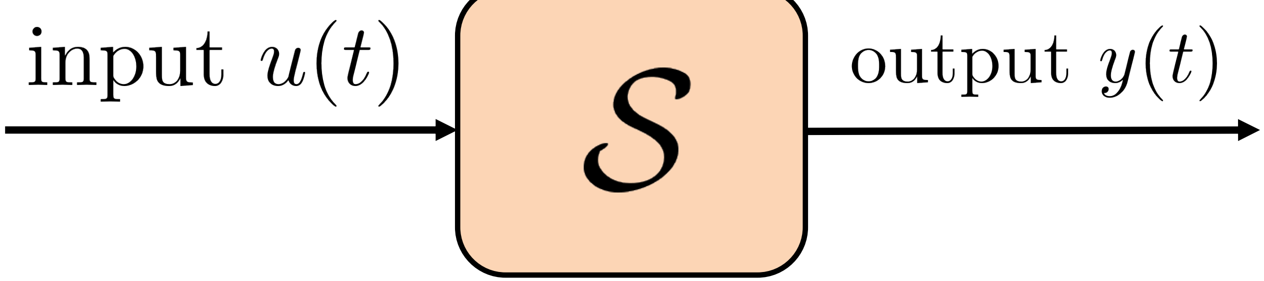
Transfer function: $G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$

State model: $A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$ $\quad B=\begin{bmatrix} b_{n-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$

$C=\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

Properties of state model

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

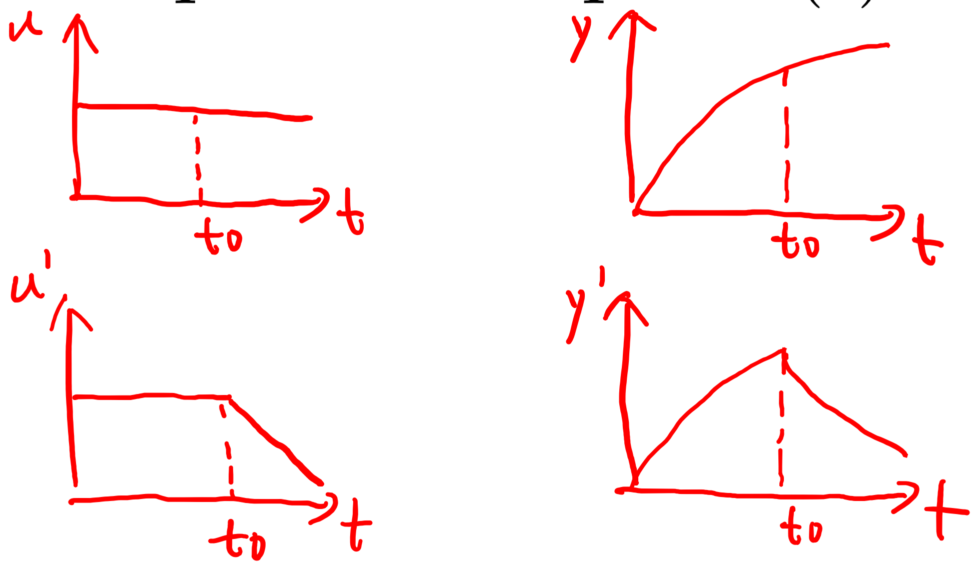


$$t \in [0, \infty)$$

Properties of state model

因果

Causality: output $y(t)$ before some time $t (\geq 0)$ does not depend on input $u(t)$ after t



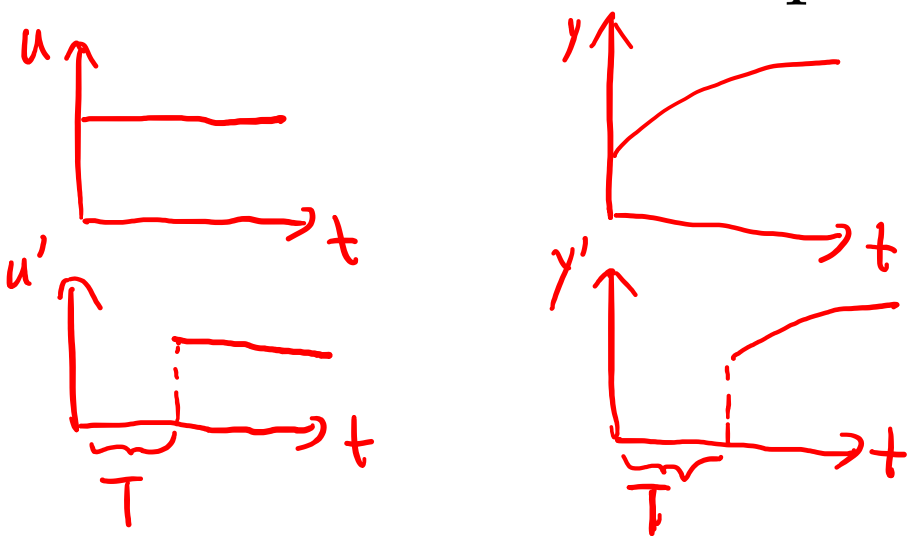
$\dot{x} = Ax + Bu$

$y = Cx$

State model is causal: let $t' \geq 0$;
then $y(t')$ depends on $x(t')$,
and $x(t')$ depends on $u(t)$ for any $t \in [0, t']$

Properties of state model

Time-invariance: time shift of input $u(t)$ to $u(t+T)$ (for some finite $T \geq 0$) results in time shift of output $y(t)$ to $y(t+T)$



State model is time-invariant: matrices A, B, C have constant entries

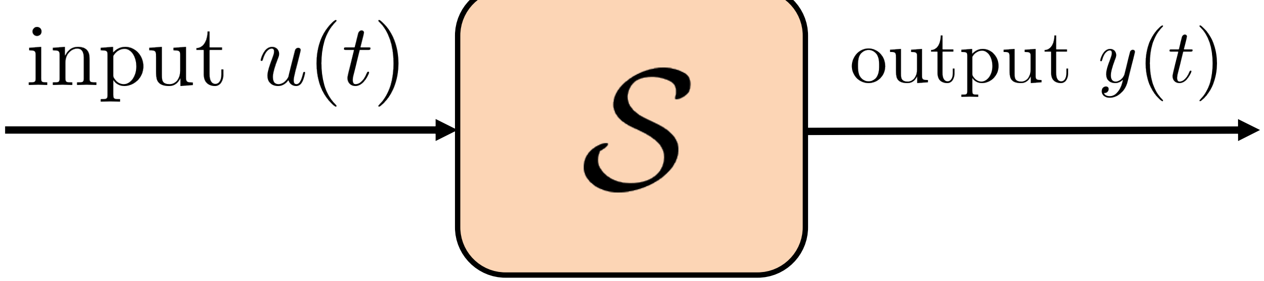
Properties of state model

Linearity: output $y(t)$ is a linear function of input $u(t)$

State model is linear:
$y(t)$ is a linear function of $x(t)$,
and $x(t)$ is a linear function of $u(t)$

Properties of state model

$$\dot{x} = Ax + Bu$$
$$y = Cx$$



$$t \in [0, \infty)$$

is causal, time-invarient, and linear (aka. LTI)