State estimation problem

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
with unknown initial state $x(0)$
Want to estimate $x(t)$, based on $y(0), \ldots, y(t)$

State estimation problem

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
with unknown initial state $x(0)$
Want to estimate $x(t)$, based on $y(0), \ldots, y(t)$

To motivate, consider the simplest state estimator:
$\dot{\hat{x}} = A\hat{x} + Bu$ ($\hat{x}$ is estimate of x)
$\hat{y} = C\hat{x}$ ($\hat{y}$ is estimate of y; y is observed but not used)

State estimation problem

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
with unknown initial state $x(0)$
Want to estimate $x(t)$, based on $y(0), \ldots, y(t)$

To motivate, consider the simplest state estimator:
$\dot{\hat{x}} = A\hat{x} + Bu$ ($\hat{x}$ is estimate of x)
$\hat{y} = C\hat{x}$ ($\hat{y}$ is estimate of y; y is observed but not used)

Define the state estimation error $e := \hat{x} - x$
Then $\dot{e} := \dot{\hat{x}} - \dot{x} = (A\hat{x} + Bu) - (Ax + Bu)$

$$= A(\hat{x} - x) + \cancel{(Bu - Bu)}$$

$$= Ae$$

State estimation problem

Now consider a revised state estimator:

$$\boxed{\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(\hat{y} - y)\\ \hat{y} &= C\hat{x}\end{aligned}}$$

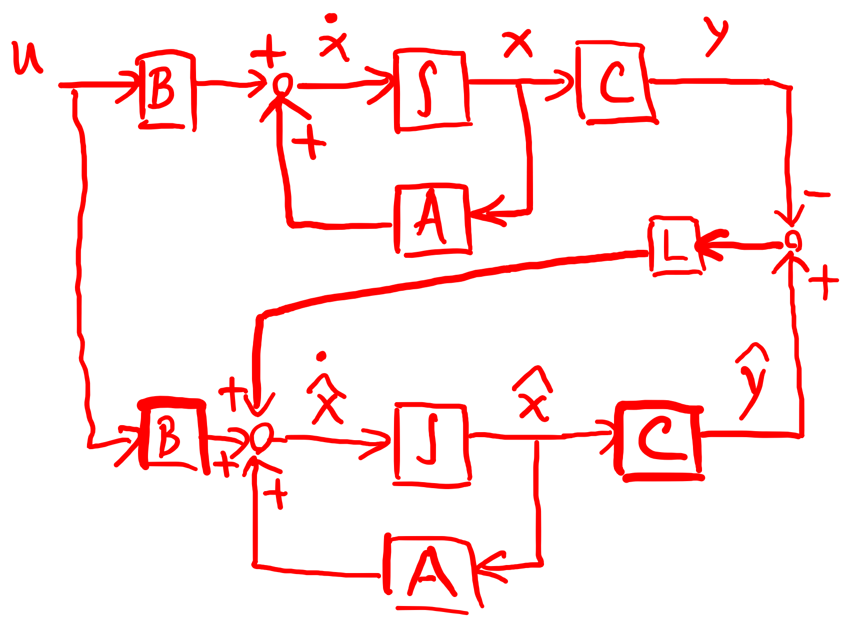
state estimator

Plant

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}}$$

$L(\hat{y} - y)$: correction proportional to output estimation error

Block diagram:



State estimation problem

Now consider a revised state estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y)$$

$$\hat{y} = C\hat{x}$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

$$= (A + LC)\hat{x} + Bu - Ly$$

State estimation problem

Now consider a revised state estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y)$$
$$\hat{y} = C\hat{x}$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$
$$= (A + LC)\hat{x} + Bu - Ly$$

$(y = Cx)$

Consider the state estimation error $e := \hat{x} - x$

Then $\boxed{\dot{e}} := \dot{\hat{x}} - \dot{x} = [(A+LC)\hat{x} + Bu - Ly] - [Ax + Bu]$

$= (A+LC)\hat{x} - LCx - Ax$

$= (A+LC)(\hat{x} - x)$

$\boxed{= (A+LC)e}$

State estimation problem

$\dot{e} = (A + LC)e$ "error system"

So if we choose L s.t. $(A + LC)$ is stable then $e(t) \to 0$ as $t \to \infty$

State estimation problem

$$\dot{e} = (A + LC)e$$

So if we choose L s.t. $(A + LC)$ is stable
then $e(t) \to 0$ as $t \to \infty$

(A^T, C^T) is stabilizable

$\Updownarrow$

This is possible exactly when (C, A) is detectable

Conclusion: state estimation problem is solvable
iff (C, A) is detectable

Detectability

For $\dot{x} = Ax + Bu,\ y = Cx$

(C, A) is *detectable* iff (A^T, C^T) is stabilizable

iff $(\exists L) A + LC$ is stable

iff $(\forall i = 1, \ldots, n)\ \mathrm{Re}(\lambda_i) \geq 0 \Rightarrow \mathrm{rank}\begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n$

Detectability

For $\dot{x} = Ax + Bu,\ y = Cx$

(C, A) is *detectable* iff (A^T, C^T) is stabilizable

iff $(\exists L) A + LC$ is stable

iff $(\forall i = 1, \ldots, n)\ \mathrm{Re}(\lambda_i) \geq 0 \Rightarrow \mathrm{rank} \begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n$

If (C, A) is observable, then (C, A) is *detectable*

Example

$n=2$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Is (C, A) detectable?

PBH: eigs of A: $0, 0$

$$\text{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \hline 1 & 0 \end{bmatrix} = 2$$

observable: $\text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$

(C, A) observable $\Rightarrow$ (C, A) detectable

State feedback control $u = Fx$

State-estimator output feedback control

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$ and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$ for all (unknown) initial state $x(0)$

State-estimator output feedback control

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

Consider estimator-based output feedback control:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y) \qquad L \text{ is s.t. } A + LC \text{ stable}$$

$$u = F\hat{x} \qquad F \text{ is s.t. } A + BF \text{ stable}$$

output

$\dot{x} = Ax + Bu$
$y = Cx$

State-estimator output feedback control

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

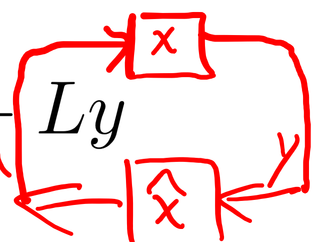
Consider estimator-based output feedback control:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y) \qquad L \text{ is s.t. } A + LC \text{ stable}$$

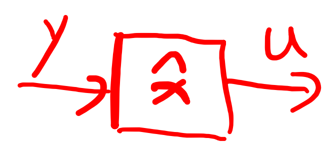
$$u = F\hat{x} \qquad F \text{ is s.t. } A + BF \text{ stable}$$

$$\dot{\hat{x}} = (A + BF + LC)\hat{x} - Ly$$

$$u = F\hat{x}$$



y: input
u: output
$\hat{x}$: state

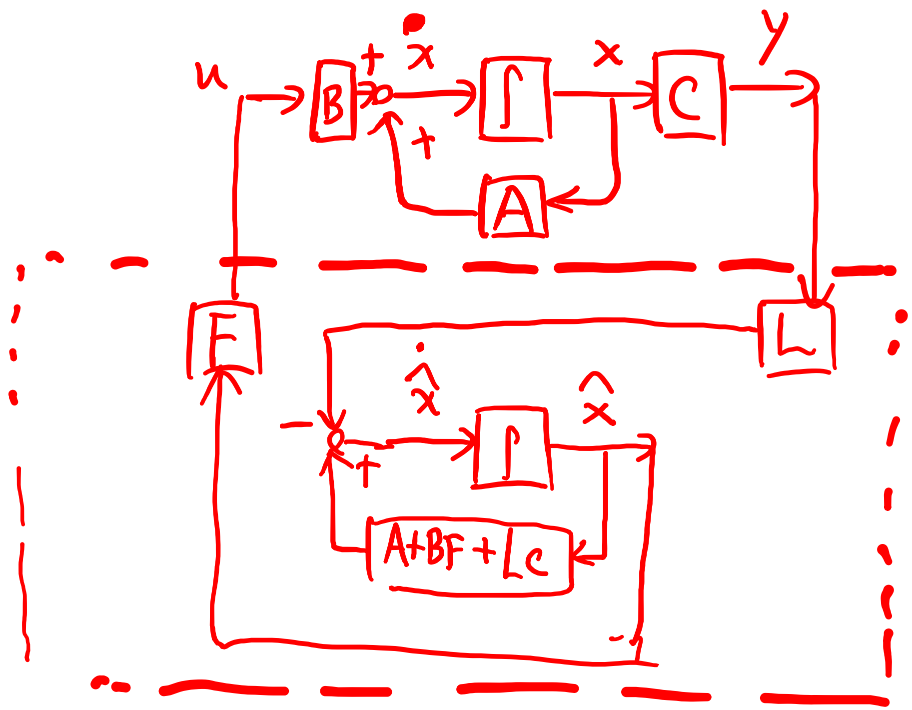


State-estimator output feedback control

Plant: $\dot{x} = Ax + Bu,\ y = Cx$

Controller:
$\dot{\hat{x}} = (A + BF + LC)\hat{x} - Ly$
$u = F\hat{x}$

Block diagram



output feedback controller

State-estimator output feedback control

Analysis:

$$\dot{x} = Ax + Bu = Ax + BF\hat{x}$$

$$\dot{\hat{x}} = (A + BF + LC)\hat{x} - LCx$$

State-estimator output feedback control

Analysis:

$$\dot{x} = Ax + Bu = Ax + BF\hat{x}$$

$$\dot{\hat{x}} = (A + BF + LC)\hat{x} - LCx$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

"closed-loop system": plant + controller

Is this closed-loop system stable?

State-estimator output feedback control

Analysis:

$$\dot{x} = Ax + Bu = Ax + BF\hat{x}$$

$$\dot{\hat{x}} = (A + BF + LC)\hat{x} - LCx$$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Is this closed-loop system stable?

Let's analyze the eigenvalues of $\begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix}$

State-estimator output feedback control

Consider a similarity transformation by

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \; (T^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix})$$

Then $T^{-1} \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} T =$

$$\begin{bmatrix} A+BF & BF \\ 0 & A+LC \end{bmatrix}$$

Recap

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$ and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$ for all (unknown) initial state $x(0)$

Recap

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable
Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

Solution:
Step 1: Design L s.t. $A + LC$ stable
Step 2: Design F s.t. $A + BF$ stable

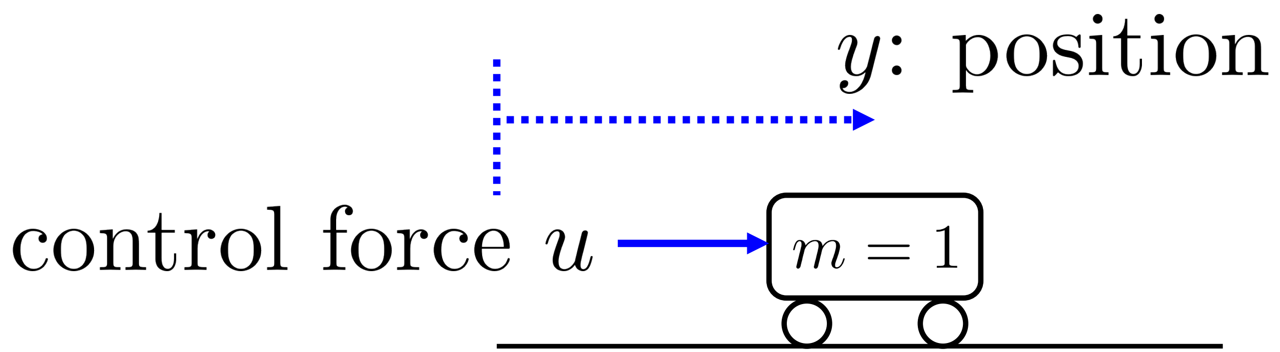
Recap

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable
Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

Solution:
Step 1: Design L s.t. $A + LC$ stable
Step 2: Design F s.t. $A + BF$ stable
Step 3: Set state-estimator based output feedback controller:
$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$
$u = F\hat{x}$

Seperation Principle

Example: self-driving car



ODE: $\ddot{y} = u$

Choose state variables: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Example: self-driving car

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Is (A, B) stabilizable? ✓

Is (C, A) detectable? ✓

Example: self-driving car

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

2×1 $\quad$ 1×2

Step 1: design L s.t. $A + LC$ has eigenvalues $-1, -2$

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

$$A + LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} L_1 & 1 \\ L_2 & 0 \end{bmatrix}$$

$$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2 \quad \cdots (1)$$

$$\left| \begin{bmatrix} \lambda - L_1 & -1 \\ -L_2 & \lambda \end{bmatrix} \right| = (\lambda - L_1)\lambda - L_2 = \lambda^2 - L_1 \lambda - L_2 \quad \cdots (2)$$

So $L_1 = -3$, $L_2 = -2$, $L = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

Example: self-driving car

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Step 2: design F s.t. $A + BF$ has eigenvalues $-1, -2$

$$F = \begin{bmatrix} -2 & -3 \end{bmatrix}$$

Example: self-driving car

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad\qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Step 3: state-estimator based output feedback controller

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A+BF+LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}}$$

$$\begin{cases} \dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} -3 \\ -2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x} - y \right) \\ u = \begin{bmatrix} -2 & -3 \end{bmatrix} \hat{x} \end{cases}$$