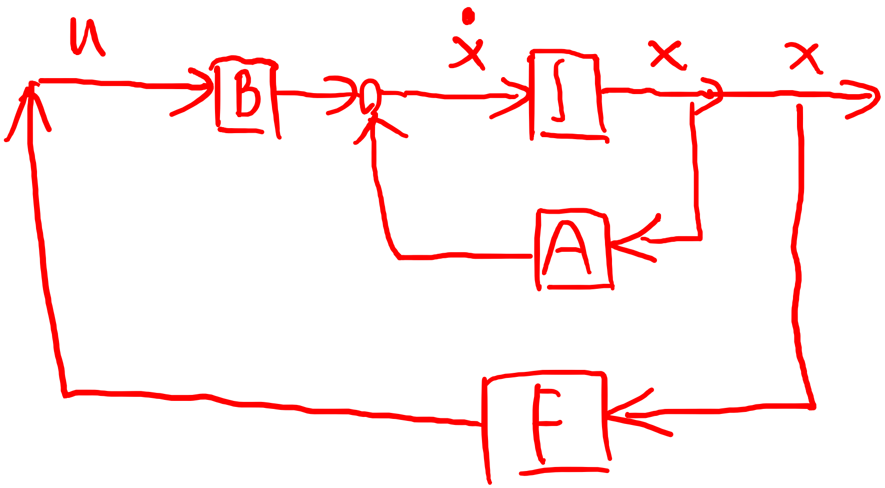
Reference Tracking

So far

$$\dot{x} = Ax + Bu$$

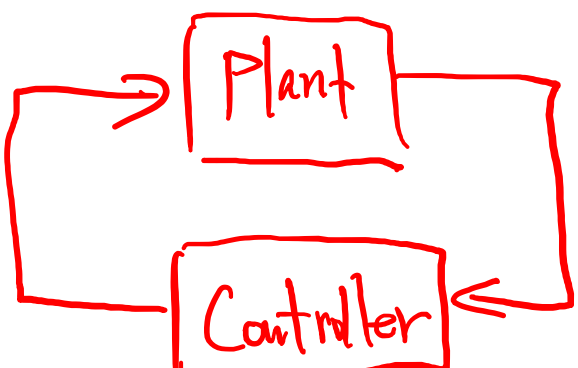
$$\underline{u} = F\underline{x}$$

Block diagram (state-feedback)



Plant (system)

Controller (static)

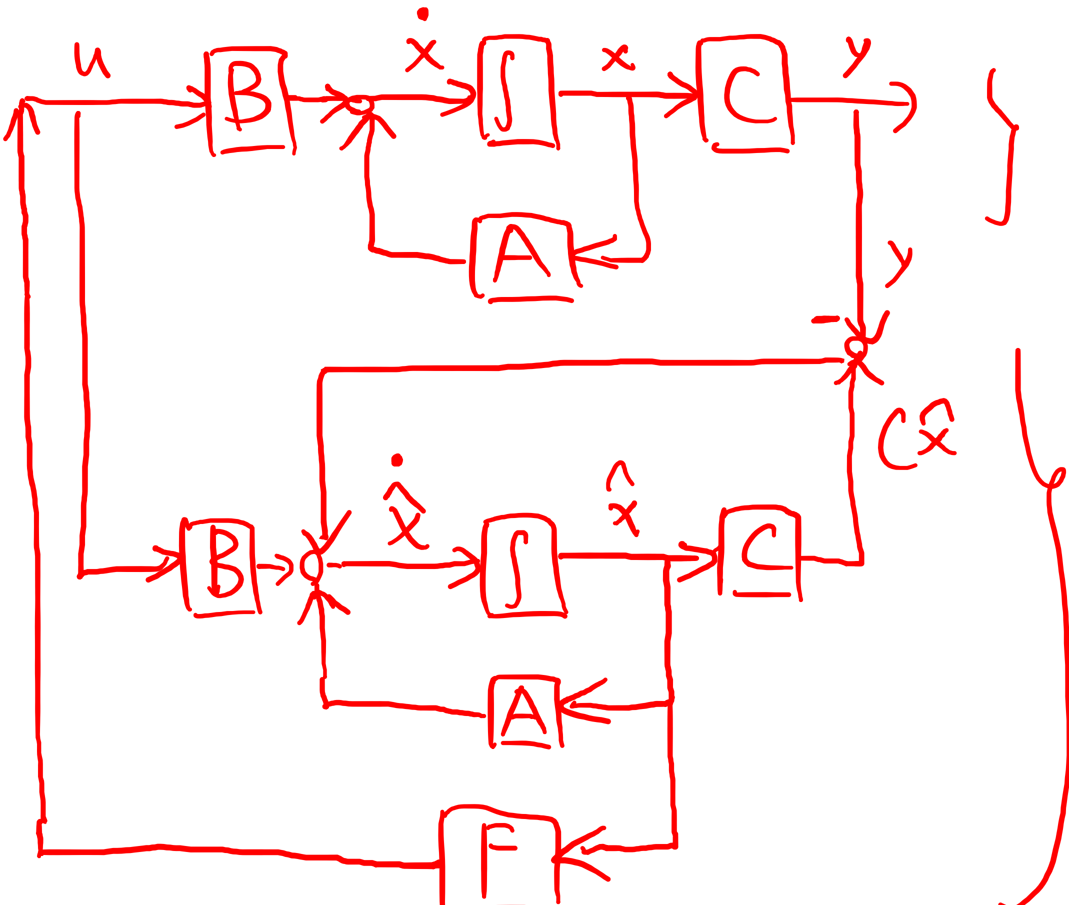


So far

Plant $\begin{cases} \dot{x} = Ax + Bu \\ \underline{\underline{y}} = Cx \end{cases}$

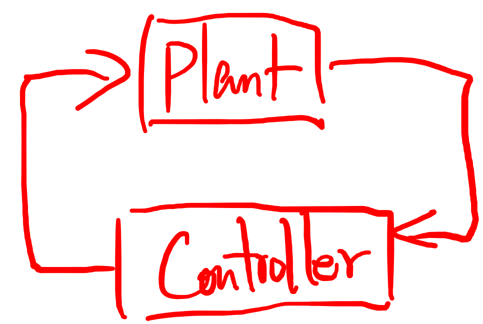
Block diagram (output-feedback)

Controller $\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y) \\ u = F\hat{x} \end{cases}$



Plant

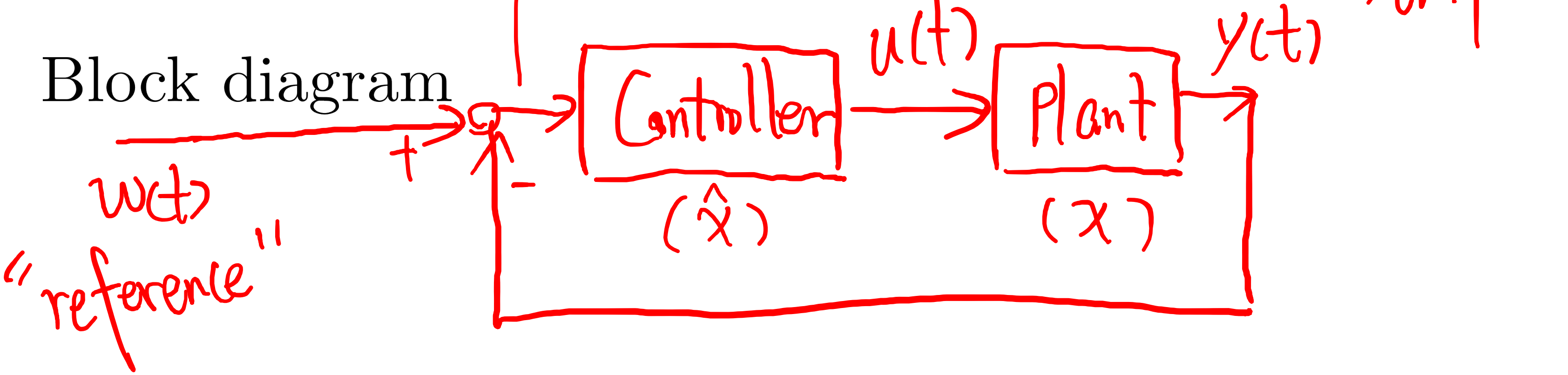
Controller (dynamic)



Reference tracking

$w(t) - y(t) =: e(t)$ "error"

Block diagram



Reference tracking

Block diagram

Reference tracking problem:
1) Closed-loop system is internally stable

2) Output $y(t)$ asymptotically tracks reference $w(t)$

Reference tracking

Reference signal $w(t) \in \mathbb{R}^k$ is generated by $\dot{w} = Sw$, where eigenvalues λ of S are unstable, i.e. $\mathrm{Re}(\lambda) \geq 0$

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1) Constant (step) signal

$w(t) = c, \quad \forall t \geq 0$

$\underline{\dot{w}} = 0 = \underline{0 \cdot w}, \quad S = 0, \quad \lambda = 0$

Reference tracking

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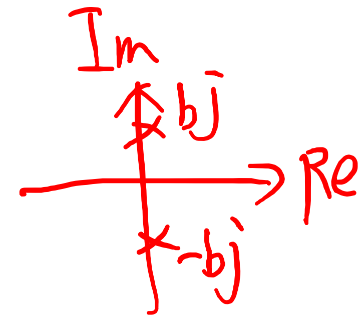
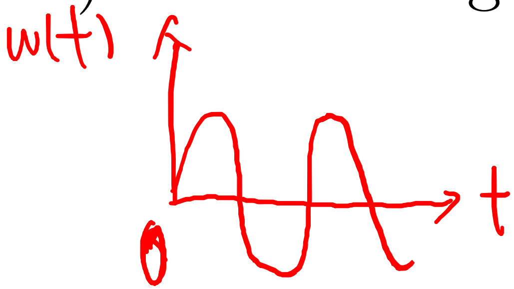
1) Constant (step) signal



2) Sinsoidal signal



$w(t) = a\sin(bt + c)$

$\ddot{\bar{w}} + b^2 \bar{w} = 0 \quad (\bar{w} \in \mathbb{R})$

$\begin{bmatrix} \bar{w} \\ \dot{\bar{w}} \end{bmatrix} = w \in \mathbb{R}^2 \qquad \dot{w} = \begin{bmatrix} 0 & 1 \\ -b^2 & 0 \end{bmatrix} w, \quad \lambda = \pm bj$

Reference tracking

Reference signal $w(t) \in \mathbb{R}^k$ is generated by $\dot{w} = Sw$, where eigenvalues λ of S are unstable, i.e. $\mathrm{Re}(\lambda) \geq 0$

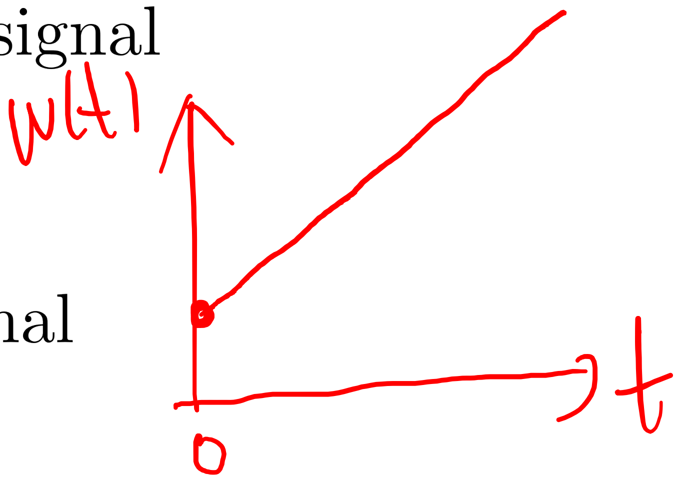
1) Constant (step) signal

$\ddot{w} = 0$

$w(t) = at+b$

$w = \begin{bmatrix} \bar{w} \\ \dot{w} \end{bmatrix} \in \mathbb{R}^2$

2) Sinsoidal signal

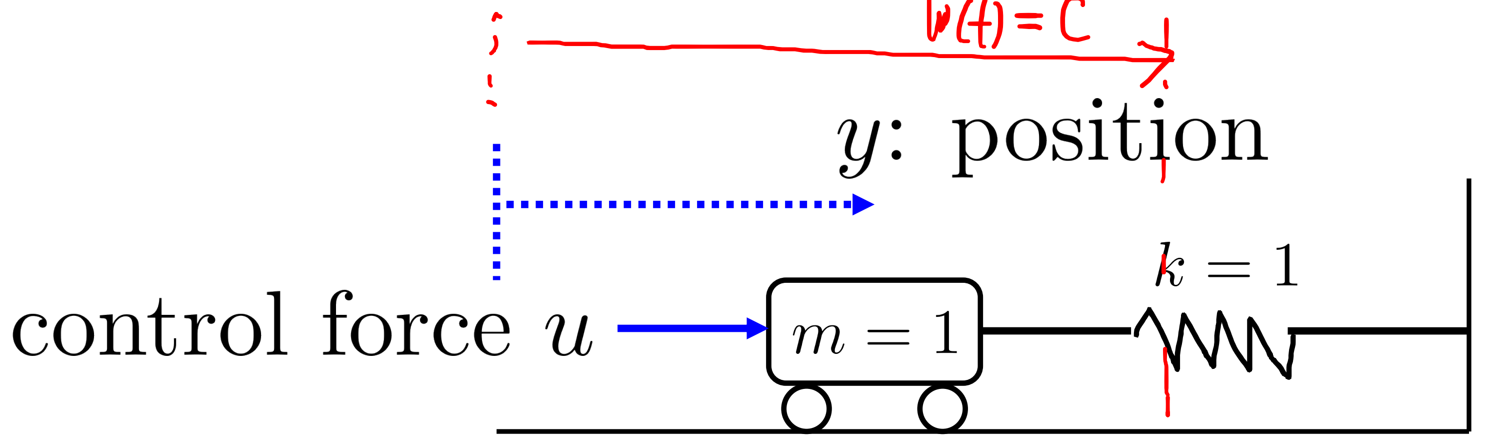


$\dot{w} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{S} w$

$\lambda_1 = \lambda_2 = 0$

3) Ramp signal

Example: car-spring system



ODE: $m\ddot{y} = u - ky$

Choose state variables: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Example: car-spring system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$y = x_1$

Suppose there is a constant reference signal $w(t) = c$ that we want the output $y(t)$ to track

So $\dot{w} = Sw$ ($S = 0$)

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Suppose there is a constant reference signal $w(t) = c$ that we want the output $y(t)$ to track

So $\dot{w} = Sw$ ($S = 0$)

Note: we do not suppose we know c, but rather the tracking error $e := w - y$

$$= Rw - Cx \qquad (R = 1)$$

Example: car-spring system

Putting all together:

$x \in \mathbb{R}^n$ $\qquad \dot{x} = Ax + Bu$ (plant)

$w \in \mathbb{R}^k$ $\qquad \dot{w} = Sw$ (reference)

$$e = Rw - Cx \quad \text{(tracking error)}$$

Consider "plant+reference":

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x \\ w \end{bmatrix}}_{\tilde{x}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} \underline{u} \quad \text{(plant + ref.)}$$

$$e = \underbrace{\begin{bmatrix} -C & R \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} x \\ w \end{bmatrix}}_{\tilde{x}}$$

Example: car-spring system

$\tilde{x} = \begin{bmatrix} x \\ w \end{bmatrix}$

Then

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

$$e = \tilde{C}\tilde{x}$$ (tracking error)

Control specifications:

1) If $(\forall t \geq 0) w(t) = 0$, then $(\forall x(0) \in \mathbb{R}^n) \lim_{t \to \infty} x(t) = 0$
   "internal stability"
2) $(\forall x(0) \in \mathbb{R}^n, \forall w(0) \in \mathbb{R}^k) \lim_{t \to \infty} e(t) = 0$
   "reference tracking"

State-feedback control

$$\text{Consider } \overset{\dot{\tilde{x}}}{\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix}} = \overset{\tilde{A}}{\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}} \overset{\tilde{x}}{\begin{bmatrix} x \\ w \end{bmatrix}} + \overset{\tilde{B}}{\begin{bmatrix} B \\ 0 \end{bmatrix}} u$$

$$e = \underset{\tilde{C}}{[-C \;\; R]} \begin{bmatrix} x \\ w \end{bmatrix}$$

and state-feedback control $u = F\tilde{x} = [F_1 \;\; F_2] \begin{bmatrix} x \\ w \end{bmatrix}$

State-feedback control

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Assume (A, B) stabilizable.

Design F_1 s.t. $A + BF_1$ is stable (eigs. λ satisfy $\mathrm{Re}(\lambda) < 0$)

F_2: to be determined

State-feedback control

The closed-loop system is:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} F_1 & F_2 \end{bmatrix}}_{} \begin{bmatrix} x \\ w \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} A+BF_1 & BF_2 \\ 0 & S \end{bmatrix}}_{} \begin{bmatrix} x \\ w \end{bmatrix}$$

State-feedback control

The closed-loop system is:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A+BF_1 & BF_2 \\ 0 & S \end{bmatrix}}\begin{bmatrix} x \\ w \end{bmatrix}$$

We wish to block-diagnolize this matrix.

Consider $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} =$. Then $T^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$

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State-feedback control

So $T^{-1}\begin{bmatrix} A+BF_1 & BF_2 \\ 0 & S \end{bmatrix} T = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}\begin{bmatrix} A+BF_1 & BF_2 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$

$$= \begin{bmatrix} A+BF_1 & BF_2 - XS \\ 0 & S \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A+BF_1 & (A+BF_1)X - XS + BF_2 \\ 0 & S \end{bmatrix}$$

Hope: choose X s.t.

$$(A+BF_1)X - XS + BF_2 = 0$$

State-feedback control

So $T^{-1}\begin{bmatrix} A + BF_1 & BF_2 \\ 0 & S \end{bmatrix} T =$

This matrix is block-diagonal iff $(A + BF_1)X - XS + BF_2 = 0$

State-feedback control

$(A + BF_1)X - XS + BF_2 = 0$ is a Sylvester equation (general form: $MX + XN + L = 0$)

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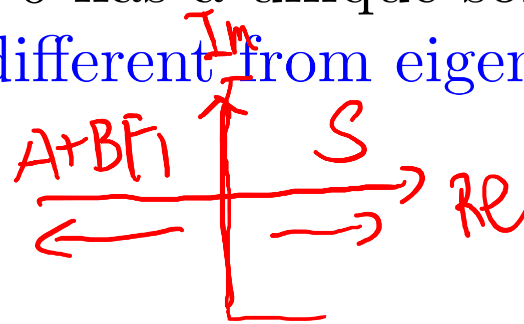
(general form: $MX + XN + L = 0$)

$\dot{w} = Sw$

$(A+BF_1)X - XS + BF_2 = 0$ has a unique solution X iff
eigenvalues of $A+BF_1$ are different from eigenvalues of S



This condition holds (why?)
So there is a unique X s.t. the above matrix is block-diagonal

State-feedback control

For $\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A + BF_1 & BF_2 \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$

$$e = [-C \;\; R] \begin{bmatrix} x \\ w \end{bmatrix}$$

and $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \cancel{=}$ where X satisfies $(A + BF_1)X - XS + BF_2 = 0$

State-feedback control

For $\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A + BF_1 & BF_2 \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$

$$e = \begin{bmatrix} -C & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

and $T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \cancel{=}$ where X satisfies $(A + BF_1)X - XS + BF_2 = 0$

Let $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T^{-1} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x - Xw \\ w \end{bmatrix}$

Then $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = T^{-1} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{T^{-1} \begin{bmatrix} A+BF_1 & BF_2 \\ 0 & S \end{bmatrix} T} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A+BF_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

State-feedback control

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A + BF_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\dot{z}_1 = (A + BF_1) z_1$

$\dot{z}_2 = S z_2$

So $\lim_{t\to\infty} z_1(t) = 0$, namely $\lim_{t\to\infty}(x(t) - Xw(t)) = 0$

Thus "internal stability" holds (why?)

Now for the tracking error

$$e = [-C \;\; R] \underbrace{T \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\begin{bmatrix} x \\ w \end{bmatrix}} = [-C \;\; R] \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = [-C \;\; -CX+R] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\Rightarrow e = (-Cz_1) + (-CX+R) z_2$

Since $z_1(t) \to 0$ and $z_2(t) \not\to 0$,
we conclude $\lim_{t\to\infty} e(t) = 0$ iff $-CX + R = 0$

"reference tracking"

State-feedback control

In total we have two matrix equations:

$$\begin{aligned}&(A + BF_1)X - XS + BF_2 = 0\\&-CX + R = 0\end{aligned}$$

The unknowns are X and F_2.

State-feedback control

In total we have two matrix equations:

$$
\begin{aligned}
&(A + BF_1)X - XS + BF_2 = 0 \\
&-CX + R = 0
\end{aligned}
$$

The unknowns are X and F_2.

Make a transformation s.t.

$$
\begin{aligned}
&AX - XS + B(F_1X + F_2) = 0 \\
&-CX + R = 0
\end{aligned}
$$

Write $U := F_1X + F_2$, then

$$
\begin{aligned}
&AX - XS + BU = 0 \\
&-CX + R = 0
\end{aligned}
$$

The unknowns are X and U.

Recap

Consider
$$\dot{x} = Ax + Bu$$
$$\dot{w} = Sw \text{ (eigenvalues of } S \text{ are unstable)}$$
$$e = Rw - Cx$$

Reference tracking problem is solvable by state-feedback control iff (A, B) is stabilizable and there exist X and U s.t.

$$\begin{aligned} AX - XS + BU &= 0 \\ -CX + R &= 0 \end{aligned} \qquad \text{(regulator equations)}$$