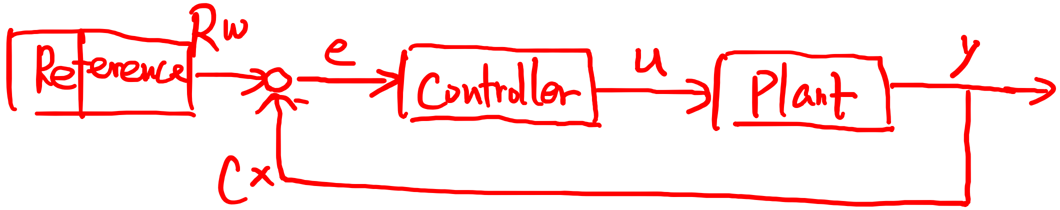
Recap



Consider $\dot{x} = Ax + Bu$ (plant)

(reference signal) $\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$ (tracking error)

Reference tracking problem is solvable by state-feedback control iff (A, B) is stabilizable and there exist X and U s.t.

1

$$AX - XS + BU = 0$$

2

$$-CX + R = 0$$

(regulator equations)

Recap

Consider
$$\dot{x} = Ax + Bu$$
$$\dot{w} = Sw \text{ (eigenvalues of } S \text{ are unstable)}$$
$$e = Rw - Cx$$

When the conditions hold, control design procedure:

1) Compute F_1 s.t. $A + BF_1$ is stable

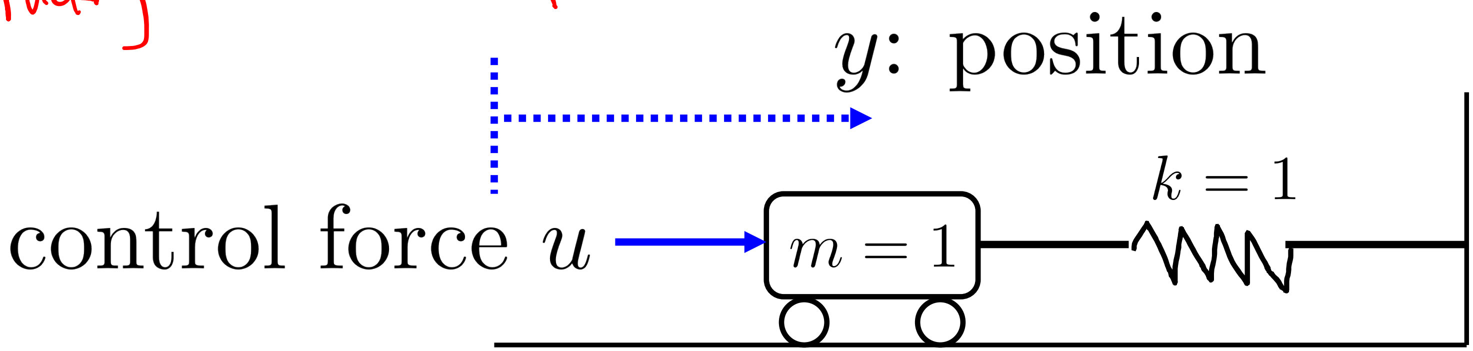
2) Solve for X, U from
$$AX - XS + BU = 0$$
$$-CX + R = 0$$

3) Set $F_2 = U - F_1 X$ ($U = F_1 X + F_2$)

4) Set $u = [F_1 \;\; F_2] \underbrace{\begin{bmatrix} x \\ w \end{bmatrix}}_{\tilde{x}}$ (state-feedback control)

Example: car-spring system

Tracking a constant reference: $\dot{w} = 0$ | error $e = w - y$ $(R=1)$



$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

(A, B) is stabilizable (P) Condition (1) ok

1) Compute F_1 s.t. eigenvalues of $A + BF_1$ are $-1, -1$

$F_1 = [0 \ \ -2]$

Example: car-spring system

Condition ?

$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 0 \end{bmatrix}$

$S=0$, $R=1$

2) Solve $AX - XS + BU = 0$

$$-CX + R = 0$$

i.e.

$$\begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cdot 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U = 0 \\ -\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + 1 = 0 \end{cases}$$

$X=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $U=1$

Example: car-spring system

3) Set $F_2 = U - F_1 X = 1 - \begin{bmatrix} 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= 1 - 0 = 1$

4) Set $u = [F_1 \;\; F_2] \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{cc:c} 0 & -2 & 1 \end{array}\right] \begin{bmatrix} x \\ w \end{bmatrix}$

State-feedback control

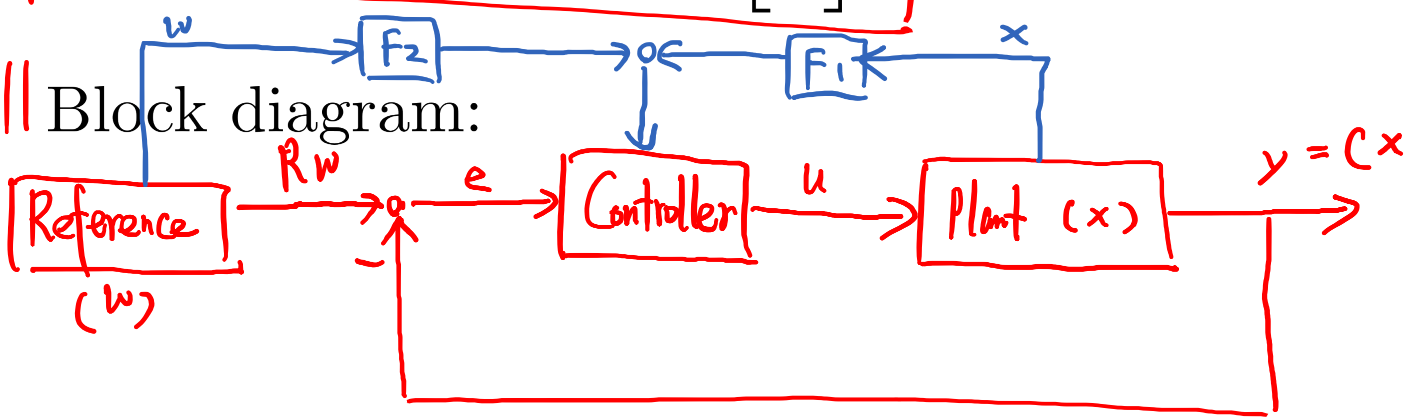
Consider $\dot{x} = Ax + Bu$

Ref. $\leftarrow$ $\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

$$\boxed{\text{Controller: } u = [F_1 \ \ F_2] \begin{bmatrix} x \\ w \end{bmatrix}} = F_1 x + F_2 w$$

Block diagram:



$\dot{x} = Ax + Bu$
$y = Cx$

$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$
$e = \tilde{C}\tilde{x}$

Output-feedback control

$$\text{Consider } \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$e = [-C \;\; R] \begin{bmatrix} x \\ w \end{bmatrix}$$

Assume only the output e (tracking error) is measurable.

Output-feedback control

Consider $\underbrace{\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix}}_{\tilde{x}} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u$

$$e = \underbrace{[-C \;\; R]}_{\tilde{C}} \begin{bmatrix} x \\ w \end{bmatrix}$$

Assume only the output e (tracking error) is measurable.

Assume $(\tilde{C}, \tilde{A})$ detectable. $\tilde{A}+L\tilde{C}$ is stable

Design an estimator-based output-feedback control:

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e)$$
$$u = F\hat{x}$$

$\hat{x}$ is estimate for $\tilde{x} = \begin{bmatrix} x \\ w \end{bmatrix}$;

Output-feedback control

Estimator-based output-feedback control:

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e)$$
$$u = \underline{F\hat{x}}$$

$$\Rightarrow \quad \dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$

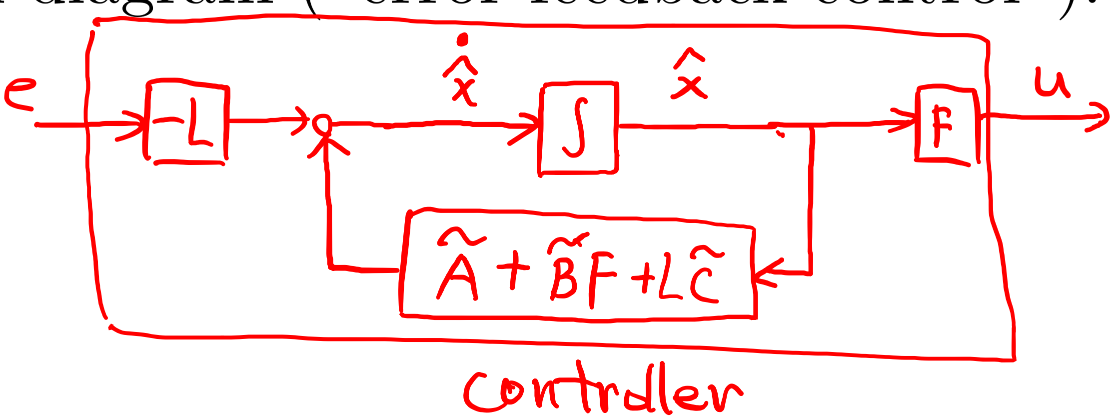
Output-feedback control

Estimator-based output-feedback control:

$$\begin{aligned}\dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e)\\ u &= F\hat{x}\end{aligned}$$

$$\Rightarrow \quad \Bigg\| \quad \begin{aligned}\dot{\hat{x}} &= (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le\\ u &= F\hat{x}\end{aligned}$$

Block diagram ("error-feedback control"):



$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} -C & R \end{bmatrix}$

Recap

Consider $\dot{x} = Ax + Bu$

$\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

Reference tracking problem is solvable by output-feedback control iff (1) (A, B) stabilizable, (2) $(\tilde{C}, \tilde{A})$ detectable, and $\exists X, U$ s.t.

$$AX - XS + BU = 0$$

(3)
$$-CX + R = 0$$

(regulator equations)

Recap

Consider $\dot{x} = Ax + Bu$

$\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

When the conditions hold, control design procedure:

1) Compute $F = [F_1 \;\; F_2]$ as in state-feedback control

2) Compute L s.t. $\tilde{A} + L\tilde{C}$ is stable

3) Set
$$\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$

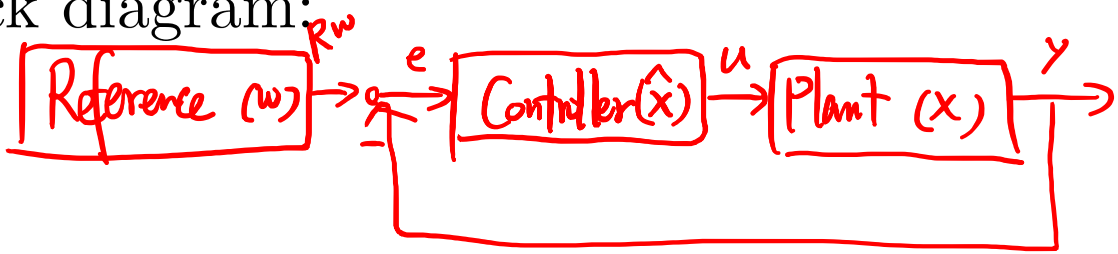
Seperation Principle

Recap

Consider
$$\dot{x} = Ax + Bu$$
$$\dot{w} = Sw \text{ (eigenvalues of } S \text{ are unstable)}$$
$$e = Rw - Cx$$

Controller:
$$\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$

Block diagram:

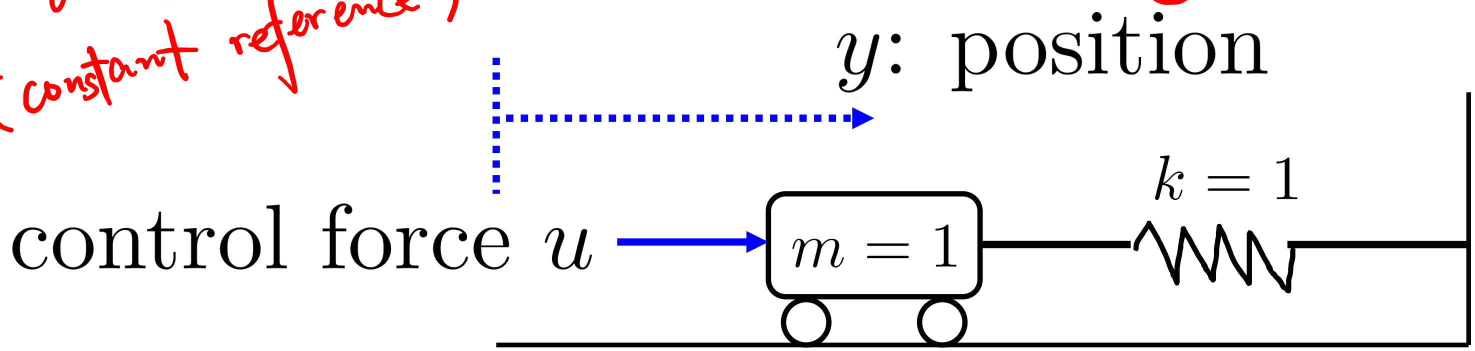


Example: car-spring system

$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$

$\tilde{C} = \begin{bmatrix} -C & R \end{bmatrix}$

$\dot{w} = 0$ (constant reference)



$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$$\tilde{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad \tilde{C} = \left[\begin{array}{cc|c} -1 & 0 & 1 \end{array}\right]$$

$(\tilde{C}, \tilde{A})$ detectable?

$$\underline{\operatorname{rank} \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \end{bmatrix} = r \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} = 3$$

Example: car-spring system

1) $\underline{F = [F_1 \ \ F_2] = [\overbrace{0 \ \ -2}^{F_1} \ \vdots \ \overbrace{1}^{F_2}]}$

2) Compute L s.t. eigenvalues of $\tilde{A} + L\tilde{C}$ are $-2, -3, -4$

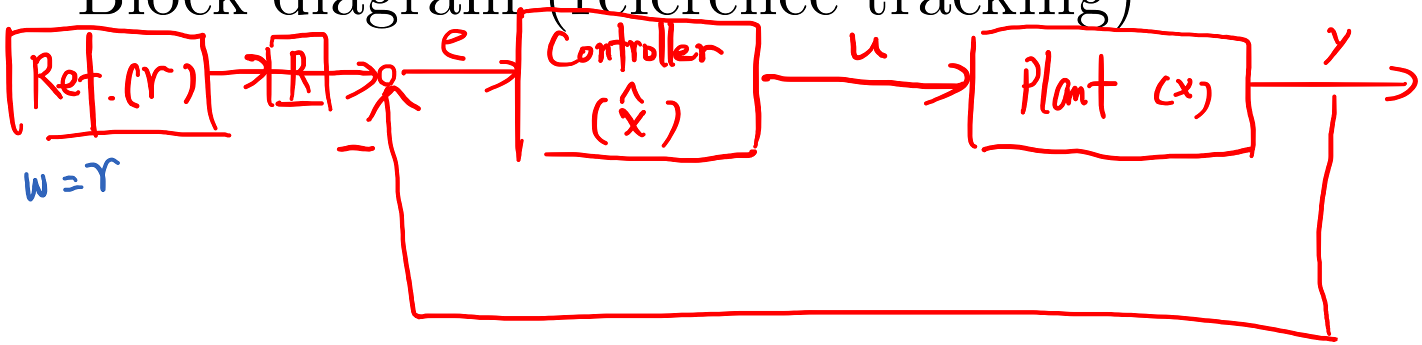
$$L = \begin{bmatrix} 15 \\ -25 \\ 24 \end{bmatrix}$$

3) Set
$$\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$

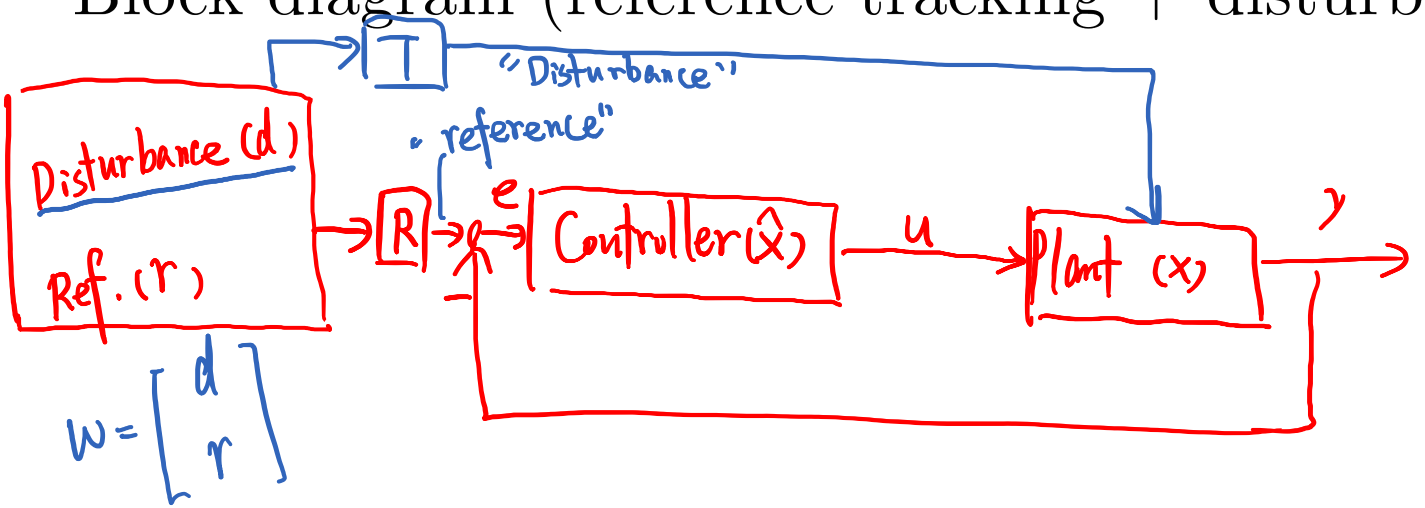
Reference Tracking
with Disturbance Rejection

So far

Block diagram (reference tracking)



Block diagram (reference tracking + disturbance rejection)



Reference tracking with disturbance rejection

w: disturbance + reference

Consider

$$\dot{x} = Ax + Bu \underbrace{+Tw}_{\text{Disturbance}}$$

$$\dot{w} = Sw$$

$$e = \underbrace{Rw}_{\text{Reference}} - Cx$$

Consider "plant+reference":

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix}}_{\tilde{x}} = \underbrace{\begin{bmatrix} A & T \\ 0 & S \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ w \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u$$

$$e = \underbrace{\begin{bmatrix} -C & R \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} x \\ w \end{bmatrix}$$

Reference tracking with disturbance rejection

Then

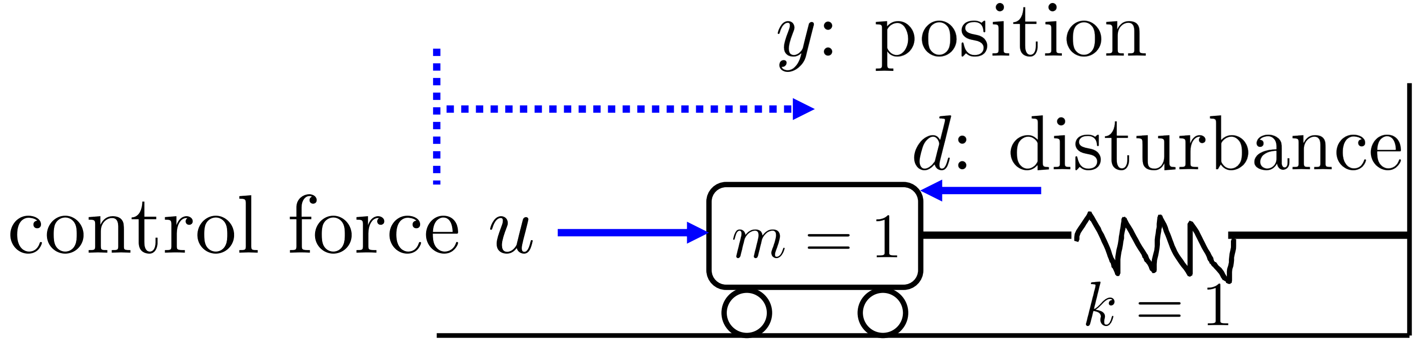
$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$e = \tilde{C}\tilde{x}$$

$$\tilde{A} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix}$$

Control specifications:

1) If $(\forall t \geq 0) w(t) = 0$, then $(\forall x(0) \in \mathbb{R}^n) \lim_{t\to\infty} x(t) = 0$
   "internal stability"
2) $(\forall x(0) \in \mathbb{R}^n, \forall w(0) \in \mathbb{R}^k) \lim_{t\to\infty} e(t) = 0$
   "reference tracking + disturbance rejection"

Example: car-spring system



$$\text{ODE: } m\ddot{y} = u - ky - d$$

$$\text{Choose state variables: } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

Example: car-spring system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \boxed{\begin{bmatrix} 0 \\ -1 \end{bmatrix} d} \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Suppose $d(t)$ is a sinusoidal disturbance of frequency 10rad/s:

$\dot{d} = S_d d$, $S_d = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix}$

Example: car-spring system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Suppose $d(t)$ is a sinusoidal disturbance of frequency 10rad/s:
$\dot{d} = S_d d$, $S_d = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix}$, $d \in \mathbb{R}^2$

Suppose there is a constant reference signal $r(t) = c$
that we want the output $y(t)$ to track: $\dot{r} = S_r r$, $S_r = 0$, $r \in \mathbb{R}$

Example: car-spring system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Suppose $d(t)$ is a sinusoidal disturbance of frequency 10rad/s: $\dot{d} = S_d d$

Suppose there is a constant reference signal $r(t) = c$ that we want the output $y(t)$ to track: $\dot{r} = S_r r$

"Exosystem": $w = \begin{bmatrix} r \\ d \end{bmatrix}$

$$\dot{w} = \begin{bmatrix} \dot{r} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} S_r & 0 \\ 0 & S_d \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -100 & 0 \end{array}\right] \begin{bmatrix} r \\ \hline d \end{bmatrix}$$

Example: car-spring system

Plant:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{T \in \mathbb{R}^{2\times 3}} w = \begin{bmatrix} r \\ d \end{bmatrix}_{3\times 1}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Tracking error:

$$e = -x_1 + r = -\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{R} w = \begin{bmatrix} r \\ d \end{bmatrix}$$

Example: car-spring system

Putting all together:

$$\dot{x} = Ax + Bu + Tw$$

$$\dot{w} = Sw$$

$$e = Rw - Cx$$

Consider "plant+reference":

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -100 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right] + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] u$$

$$e = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} w - \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Output-feedback control

$$\text{Consider} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A & T \\ 0 & S \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$e = \underbrace{[-C \ \ R]}_{\tilde{C}} \begin{bmatrix} x \\ w \end{bmatrix}$$

Assume only the output e (tracking error) is measurable.

Output-feedback control

$$\text{Consider } \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} A & T \\ 0 & S \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$e = \underbrace{[-C \;\; R]}_{\tilde{C}} \begin{bmatrix} x \\ w \end{bmatrix}$$

Assume only the output e (tracking error) is measurable.

Assume $(\tilde{C}, \tilde{A})$ detectable.
Design an estimator-based output-feedback control:

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e) \\ u &= F\hat{x} \end{aligned}$$

Output-feedback control

Estimator-based output-feedback control:

$$
\begin{aligned}
\dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e) \\
u &= F\hat{x}
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{\begin{aligned}
\dot{\hat{x}} &= (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le \\
u &= F\hat{x}
\end{aligned}}
$$

Block diagram ("error-feedback control"):

Recap

Disturbance

Consider $\dot{x} = Ax + Bu + Tw$
$\dot{w} = Sw$ (eigenvalues of S are unstable)
$e = Rw - Cx$ Tracking error

Reference tracking with disturbance rejection problem is solvable by output-feedback control
iff (A, B) stabilizable, $(\tilde{C}, \tilde{A})$ detectable, and $\exists X, U$ s.t.

1 2 3

$$AX - XS + (BU + T) = 0$$
$$-CX + R = 0$$

(regulator equations)

Recap

Consider $\dot{x} = Ax + Bu + Tw$
$\dot{w} = Sw$ (eigenvalues of S are unstable)
$e = Rw - Cx$

When the conditions hold, control design procedure:

1) Compute $F = [F_1 \;\; F_2]$ as in state-feedback control

   1.1) Compute F_1 s.t. $A + BF_1$ is stable

   1.2) Solve for X, U from $AX - XS + (BU + T) = 0$
   $-CX + R = 0$

   1.3) Set $F_2 = U - F_1 X$

Recap

Consider $\dot{x} = Ax + Bu + Tw$

$\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

When the conditions hold, control design procedure:

1) Compute $F = [F_1 \;\; F_2]$ as in state-feedback control

   1.1) Compute F_1 s.t. $A + BF_1$ is stable

   1.2) Solve for X, U from $AX - XS + (BU + T) = 0$

   $-CX + R = 0$

   1.3) Set $F_2 = U - F_1 X$

2) Compute L s.t. $\tilde{A} + L\tilde{C}$ is stable

3) Set $\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$

$u = F\hat{x}$