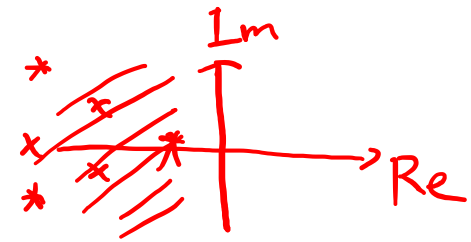
Optimal Control

Linear quadratic regulation



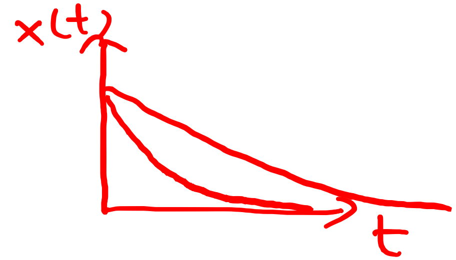
Consider $\dot{x} = Ax + Bu \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$

Suppose (A, B) is stabilizable.

Then we know we can always design F s.t. $\underline{A + BF}$ is stable, so that $(\forall x(0) \in \mathbb{R}^n) x(t) \to 0$ as $t \to \infty$

But there are infinite choices of F

Linear quadratic regulation



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But there are infinite choices of F

Want to choose F s.t.

1) $x(t)$ goes to 0 as quickly as possible (state energy)

$$\int_0^\infty ||x(t)||^2 dt$$

2) $u(t)$ stays as small as possible (control energy)

$$\int_0^\infty ||u(t)||^2 dt$$

Linear quadratic regulation

Consider $\dot{x} = Ax + Bu \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$

Suppose (A, B) is stabilizable.

Problem: Design $u = Fx$ s.t. the following is *minimized*:

$$J(x, u) = \int_0^\infty (||x(t)||^2 + \rho ||u(t)||^2) dt$$

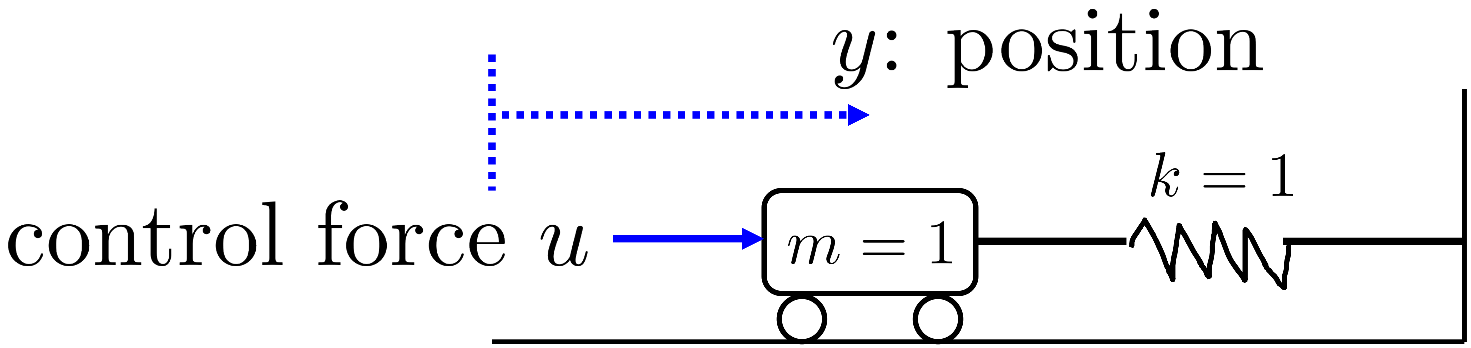
Constant $\rho (> 0)$ represents *weight*:

$\rho < 1$: state energy is more important

$\rho = 1$: state & control are equally important

$\rho > 1$: control energy is more important

Example: car-spring system



$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad\qquad (A, B) \text{ stabilizable}$$

Find a stabilizing $u = Fx$ that minimizes

$$J(x, u) = \int_0^\infty (||x(t)||^2 + \underbrace{\tfrac{1}{2}}_{\rho}||u(t)||^2)dt$$

$\|x\| = \sqrt{x^T x}$

Linear quadratic regulation

Since $||x||^2 = x^\top x = x^\top Q x$, where $Q = I$

one can consider a more general form: $x^\top Q x$
where Q is *symmetric* and *positive semidefinite*

Q is symmetric if ~~A~~ $Q^T = Q$

Q is positive semidefinite if $(\forall x \in \mathbb{R}^n) \quad x^T Q x \geq 0$

Linear quadratic regulation

Since $||x||^2 = x^\top x = x^\top Q x$, where $Q = I$

one can consider a more general form: $x^\top Q x$
where Q is *symmetric* and *positive semidefinite*

Q is symmetric if

Q is positive semidefinite if

Fact: Q is positive semidefinite iff all its eigenvalues $\lambda \geq 0$

Example

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

×

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

○

$$Q_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In fact Q_3 is *positive definite*

Linear quadratic regulation

Q is pos. semidef.

$Q \geqslant 0$

Q is positive definite if $(\forall x) x \neq 0 \Rightarrow x^\top Q x > 0$

$Q > 0$

Fact: Q is positive definite iff all its eigenvalues $\lambda > 0$

Fact: If Q is positive definite, then Q^{-1} exists and $Q^{-1} > 0$

Linear quadratic regulation

Consider $\dot{x} = Ax + Bu \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$

Suppose (A, B) is stabilizable.

Let $Q > 0$, $R > 0$, and consider the quadratic cost function

$$J(x,u) = \int_0^\infty \left(\underbrace{x(t)^\top Q x(t)}_{\|x(t)\|^2} + \underbrace{u(t)^\top R u(t)}_{\rho \cdot \|u(t)\|^2}\right) dt$$

$\|x(t)\|^2 = x^\top(t)\,\underbrace{I}_{Q} \cdot x(t)$

$\rho \cdot \|u(t)\|^2 = u^\top(t)\,\underbrace{(\rho \cdot I)}_{R}\,u(t)$

Linear quadratic regulation

Consider $\dot{x} = Ax + Bu \quad (x \in \mathbb{R}^n,\, u \in \mathbb{R}^m)$

Suppose (A, B) is stabilizable.

Let $Q > 0$, $R > 0$, and consider the quadratic cost function

$$J(x, u) = \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t))dt$$

Linear Quadratic Regulation (LQR) Problem:

Design a state-feedback control $u = Fx$ s.t.

1) $(\forall x(0)) x(t) \to 0$ as $t \to \infty$
2) $J(x, u)$ is minimized

Linear quadratic regulation

Consider $H(x,u) = -\int_0^\infty \big((Ax+Bu)^\top Px + x^\top P(Ax+Bu)\big)\,dt$ where P is a symmetric matrix.

$$
\begin{aligned}
H(x,u) &= -\int_0^\infty \big((\underline{Ax+Bu})^\top Px + x^\top P(\underline{Ax+Bu})\big)\,dt \\
&= -\int_0^\infty (\dot{x}^\top Px + x^\top P\dot{x})dt \\
&= -\int_0^\infty (\tfrac{d}{dt}(x^\top Px))dt \\
&= -(x^\top Px)|_0^\infty = x(0)^\top Px(0) - \lim_{t\to\infty} x(t)^\top Px(t) \\
&= x(0)^\top Px(0) \qquad (\text{if } x(t)\to 0 \text{ as } t\to\infty)
\end{aligned}
$$

So $H(x,u)$ depends only on $x(0)$ (if $x(t)\to 0$ as $t\to\infty$)

Linear quadratic regulation

Consider $H(x,u) = -\int_0^\infty \left((Ax+Bu)^\top Px + x^\top P(Ax+Bu)\right) dt$ where P is a symmetric matrix.

$x^T A^T P x + u^T B^T P x$

$x^T P A x + x^T P B u$

$J(x,u) = \int_0^\infty (x^\top Q x + u^\top R u) dt$

$= H(x,u) - H(x,u) + \int_0^\infty (x^\top Q x + u^\top R u) dt$

$= H(x,u) + \int_0^\infty (x^\top Q x + u^\top R u + (Ax+Bu)^\top Px + x^\top P(Ax+Bu)) dt$

$= H(x,u) + \int_0^\infty (x^\top (A^\top P + PA + Q)x + u^\top R u + 2u^\top B^\top P x) dt$

$(u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x)$

$= u^T R u + 2u^T B^T P x + x^T P B R^{-1} B^T P x$

$x \cdot a \cdot x + u \cdot b \cdot u + 2 u \cdot c \cdot x$

$= a x^2 + 2c \cdot xu + b u^2$

$= (x - du)^2 + f u^2$

Linear quadratic regulation

Consider $H(x,u) = -\int_0^\infty \big((Ax+Bu)^\top Px + x^\top P(Ax+Bu)\big)\,dt$
where P is a symmetric matrix.

$$J(x,u) = \int_0^\infty (x^\top Qx + u^\top Ru)dt$$

$$= H(x,u) - H(x,u) + \int_0^\infty (x^\top Qx + u^\top Ru)dt$$

$$= H(x,u) + \int_0^\infty (x^\top Qx + u^\top Ru + (Ax+Bu)^\top Px + x^\top P(Ax+Bu))dt$$

$$= H(x,u) + \int_0^\infty (x^\top (A^\top P + PA + Q)x + u^\top Ru + 2u^\top B^\top Px)dt$$

$$= H(x,u) + \int_0^\infty (x^\top (A^\top P + PA + Q - PBR^{-1}B^\top P)x$$
$$+x^\top PBR^{-1}B^\top Px + u^\top Ru + 2u^\top B^\top Px)dt$$

$$= H(x,u) + \int_0^\infty (\underline{x^\top (A^\top P + PA + Q - PBR^{-1}B^\top P)x}$$
$$+\underline{(u + R^{-1}B^\top Px)^\top R(u + R^{-1}B^\top Px)})dt$$

So if $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$ and $u = (-R^{-1}B^\top P)x$
then $J(x,u) = H(x,u)$

Linear quadratic regulation

Consider $H(x,u) = -\int_0^\infty \left((Ax+Bu)^\top Px + x^\top P(Ax+Bu)\right) dt$
where P is a symmetric matrix.

$$J(x,u) = \int_0^\infty (x^\top Qx + u^\top Ru)dt$$
$$= H(x,u) + \int_0^\infty (x^\top (A^\top P + PA + Q - PBR^{-1}B^\top P)x$$
$$+(u + R^{-1}B^\top Px)^\top R(u + R^{-1}B^\top Px))dt$$

So if $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$ and $u = \underbrace{(-R^{-1}B^\top P)}_{F}x$
then $J(x,u) = H(x,u)$

Moreover, since $\dot{x} = Ax + Bu = (\underbrace{A - R^{-1}B^\top P}_{A+BF})x$
if $A - R^{-1}B^\top P$ is stable, then $x(t) \to 0$ for all $x(0)$
and $J(x,u) = x(0)^\top Px(0)$ (minimized)

Recap

Consider $\dot{x} = Ax + Bu \quad (x \in \mathbb{R}^n,\ u \in \mathbb{R}^m)$

Suppose (A, B) is stabilizable.

Let $Q > 0$, $R > 0$, and consider the quadratic cost function

$$J(x,u) = \int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t))dt$$

symmetric

If we can find P s.t. $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$
and $A - R^{-1}B^\top P$ is stable

then $u = Fx$, $F = -R^{-1}B^\top P$, solves LQR problem, i.e.

1) $(\forall x(0))x(t) \to 0$ as $t \to \infty$
2) $J(x,u)$ is minimized

Recap

Fact:

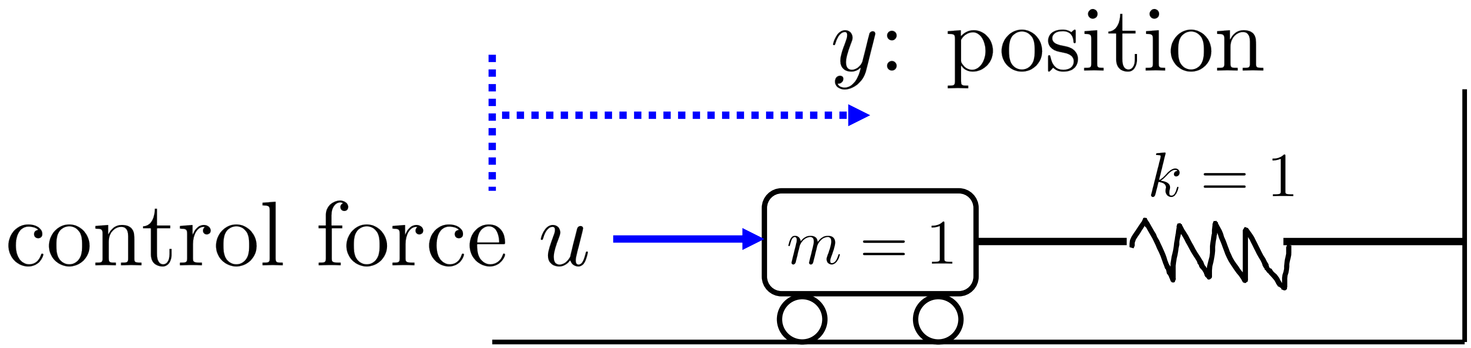
If (A, B) is stabilizable,
there always exists a (symmetric) positive definite P s.t.
$A^\top P + PA + Q - PBR^{-1}B^\top P = 0$
and $A - R^{-1}B^\top P$ is stable

Theorem:
LQR problem is solved by $u = Fx$, $F = -R^{-1}B^\top P$ where

$$A^\top P + PA + Q - PBR^{-1}B^\top P = 0 \quad (*)$$

Solution Procedure:
step 1: Solve P from $(*)$
step 2: $u = Fx$, $F = -R^{-1}B^T \underline{\underline{P}}$

Example: car-spring system



$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

(A, B) stabilizable

Find a stabilizing $u = Fx$ that minimizes

$J(x, u) = \int_0^\infty (||x(t)||^2 + \frac{1}{2}||u(t)||^2)dt$

$Q = I$ and $R = 1/2$

Example: car-spring system

Algebraic Riccati equation

Find a (symmetric) positive definite P s.t.

$$\boxed{A^\top P + PA + Q - PBR^{-1}B^\top P = 0}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -p_2 & -p_3 \\ p_1 & p_2 \end{bmatrix} + \begin{bmatrix} -p_2 & p_1 \\ -p_3 & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} p_2 & p_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2p_2+1 & -p_3+p_1 \\ p_1-p_3 & 2p_2+1 \end{bmatrix} + 2 \cdot \begin{bmatrix} p_2^2 & p_2p_3 \\ p_2p_3 & p_3^2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2p_2+1+2p_2^2 & -p_3+p_1+2p_2p_3 \\ -p_3+p_1+2p_2p_3 & 2p_2+1+2p_3^2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -2p_2+1+2p_2^2 = 0 \\ -p_3+p_1+2p_2p_3 = 0 \\ 2p_2+1+2p_3^2 = 0 \end{cases} \quad \cdots \Rightarrow \begin{cases} p_1 = 1.61 \\ p_2 = 0.37 \\ p_3 = 0.93 \end{cases}$$

$$P = \begin{bmatrix} 1.61 & 0.37 \\ 0.37 & 0.93 \end{bmatrix}$$

$u = Fx$,

$$F = -2\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1.61 & 0.37 \\ 0.37 & 0.93 \end{bmatrix} = \begin{bmatrix} -0.73 & -1.86 \end{bmatrix}$$

$$A+BF = \begin{bmatrix} 0 & 1 \\ -1.73 & -1.86 \end{bmatrix}$$

eigs: $-0.93 \pm 0.93j$