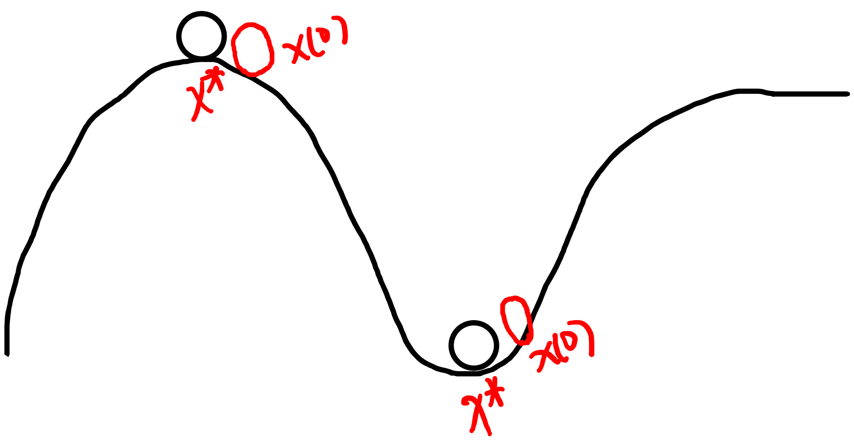
Stability

Intuition of stability

- Stability of a dynamic system (work of Russian mathematician A.M. Lyapunov, 1892)



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

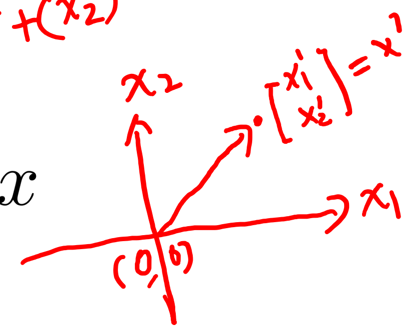
Stability of linear state model

Consider $\dot{x} = Ax$, and the equilibrium point $x^* = 0$

(x^* may be stable or unstable)

First, we need to define what is x^* being stable

$\|x'\|_2 = \sqrt{(x')^T(x')} = \sqrt{(x_1')^2 + (x_2')^2}$



$x(0)$

Let $x(0)$ be an initial state of $\dot{x} = Ax$

$x(0)$ $\|x(0)\|$ small

If $x(0)$ is near the origin ($x^* = 0$),

does $x(t)$ remain near the origin for all $t \geq 0$?

$\|x(t)\|$ small

What mathematical object means 'near the origin'?

Stability of linear state model

$\forall$: for all

$\exists$: exist

Defn. Consider $\dot{x} = Ax$ and the origin $x^* = 0$.
The origin is *stable* if

$\forall \epsilon$

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x(0) \in \mathbb{R}^n)||x(0)|| < \delta$$
$$\Rightarrow (\forall t > 0)||x(t)|| < \epsilon$$

imply (if ... then..)

logic formula

Aside: logic

propositional

Propositional logic:

negation, conjunction, disjunction, implication

negation

truth table:

		"not A"	"and"	"or"	"A implies B"	
A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$(\neg A) \vee B$
0	0	1	0	0	1	1
0	1	1	0	1	1	1
1	0	0	0	1	0	0
1	1	0	1	1	1	1

Aside: logic

First-order

First-order logic:

universal quantifier $\forall$, exitential quantifier $\exists$

e.g. $(\exists N > 0)(\forall n \geq 1)|a_n| \leq N$

1) Each variable n, N must be quantified by $\forall$ or $\exists$

2) Order of $\forall$, $\exists$ is crucial

$$(\forall n \geq 1)(\exists N > 0)|a_n| \leq N$$

3) Negation of a first-order logical statement

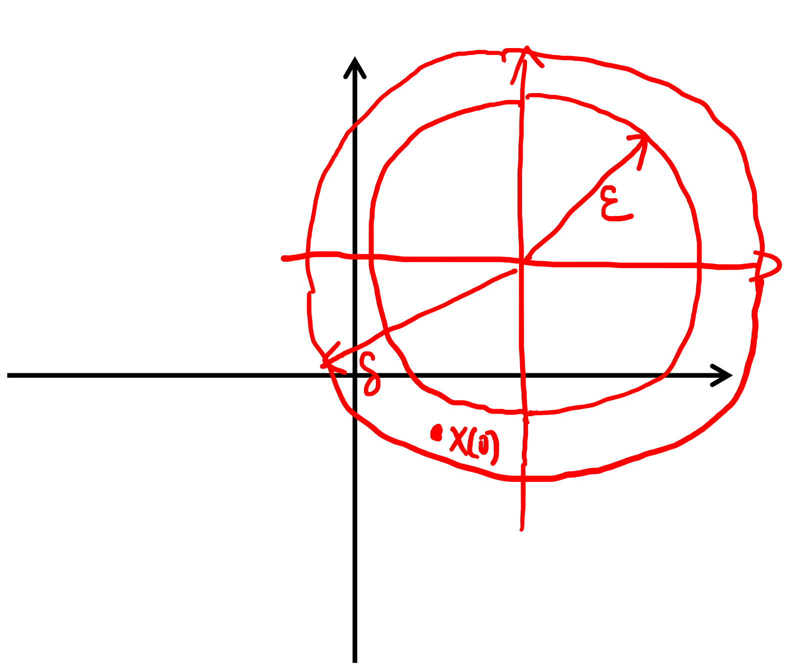
$$\neg(\exists N > 0)(\forall n \geq 1)|a_n| \leq N$$

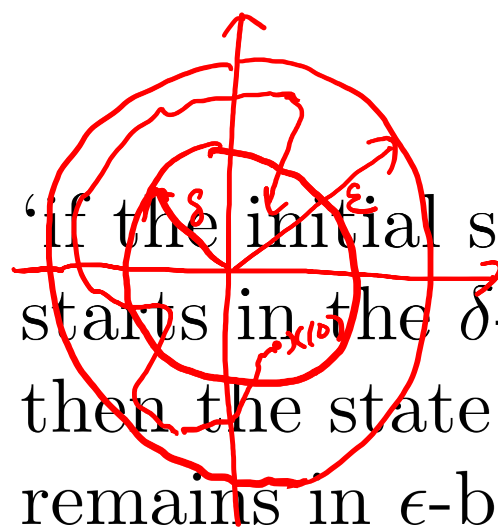
$\Leftrightarrow (\forall N > 0)(\exists n \geq 1) \underline{\neg(|a_n| \leq N)}$ $\quad |a_n| > N$

Stability of linear state model

Defn. Consider $\dot{x} = Ax$ and the origin $x^* = 0$.
The origin is *stable* if

$$(\forall\epsilon \overset{\forall\varepsilon}{>} 0)(\exists\delta_{(\varepsilon)} > 0)(\forall x(0) \in \mathbb{R}^n)||x(0)|| < \delta$$
$$\Rightarrow (\forall t > 0)||x(t)|| < \epsilon$$





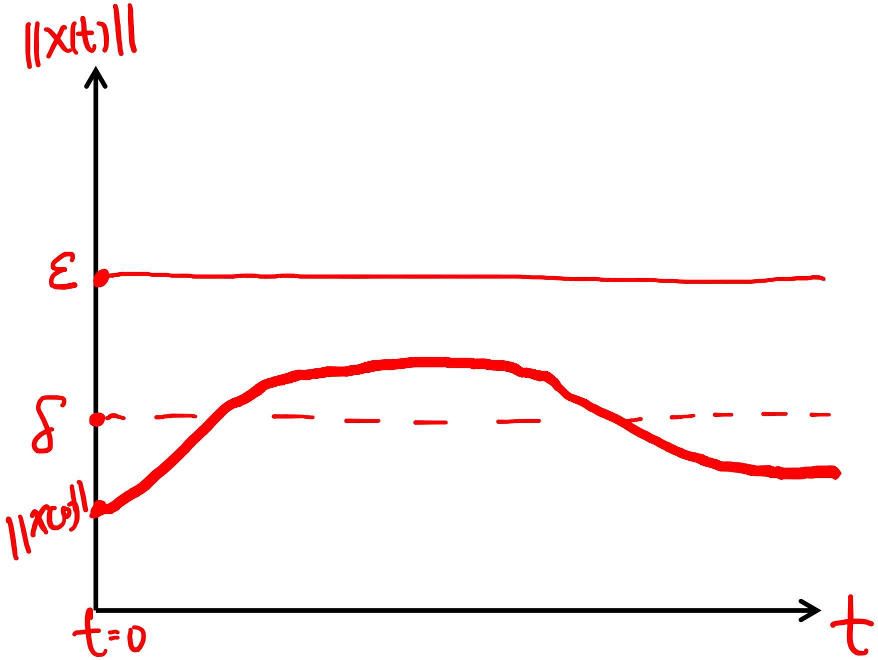
'if the initial state starts in the δ-ball, then the state remains in ϵ-ball'

Stability of linear state model

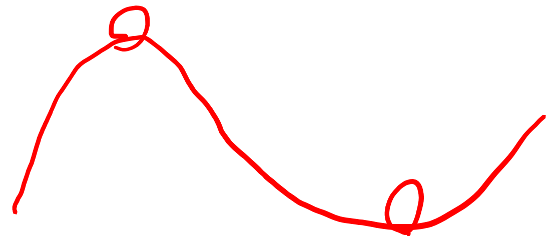
Defn. Consider $\dot{x} = Ax$ and the origin $x^* = 0$.
The origin is *stable* if

$\forall\varepsilon$

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x(0) \in \mathbb{R}^n) ||x(0)|| < \delta$$
$$\Rightarrow (\forall t > 0) ||x(t)|| < \epsilon$$

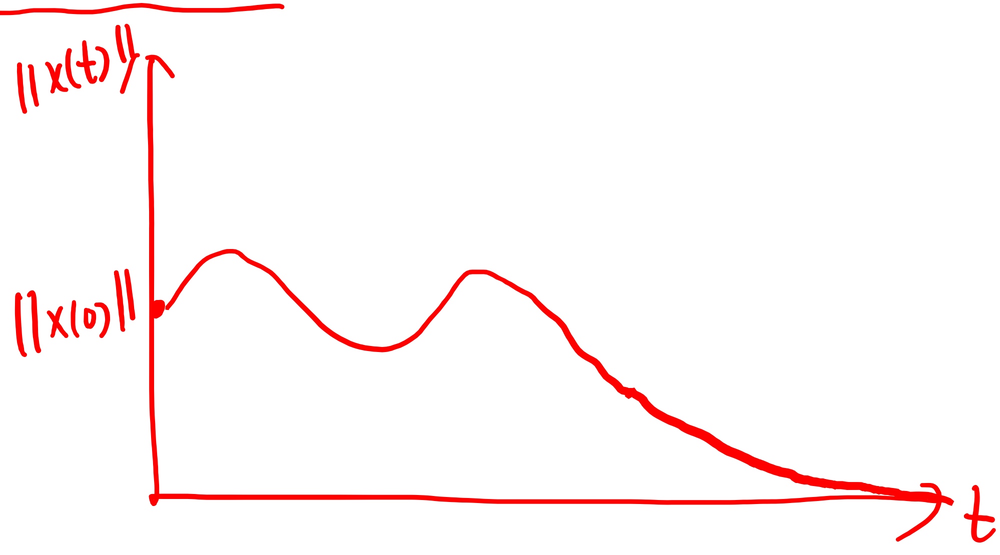


Asymptotic stability



Defn. Consider $\dot{x} = Ax$ and the origin $x^* = 0$. The origin is *aymptotically stable* if

1) it is stable, and 1
2) $\lim_{t\to\infty} x(t) = 0$ 2



Stability & asymptotic stability

1) asymptotic stability implies stability,
but the reverse need not be true

2) Stability means influence of $x(0)$ remains bounded;
asymptotic stability means influence of $x(0)$ is irrelevant

Unstable

Unstable

Defn. Consider $\dot{x} = Ax$ and the origin $x^* = 0$.
The origin is *unstable* if it is not stable, i.e.

$$\neg \, (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x(0)) \, \|x(0)\| < \delta \Rightarrow (\forall t > 0) \, \|x(t)\| < \varepsilon$$

How to determine stability

Given $\dot{x} = Ax$, to determine if the origin $x^* = 0$ is (aymptotically) stable, we need to know $x(t)$, $t \geq 0$

Our goal:

Given $\dot{x} = Ax$ and an initial state $x(0)$, derive a formula for $x(t)$, $t \geq 0$

Initial

Initial value problem

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$) and $x(0) = x_0$,

compute $x(t) =?$

E.g. first-order system: $\dot{x} = ax$ ($a, x \in \mathbb{R}$)

and $x(0) = x_0 \in \mathbb{R}$

$x(t) = e^{at} x_0$

Solution: $x(t) = e^{at}x_0$

$\dot{x}(t) = a\underline{e^{at}x_0} = ax$

Matrix exponential

Taylor

Taylor series: $e^{at} = 1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots$

Define: $e^{At} \triangleq I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$

(matrix exponential)

Matrix exponential

e^{At}

1) $e^{At}\big|_{t=0} = I$

2) e^{At} is always invertible; $(e^{At})^{-1} = e^{-At}$

e^{At_1}

3) $e^{At_1} e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \geq 0$

$e^{A_1 t}$

$e^{A_1 t} e^{A_2 t} = e^{(A_1+A_2)t}$ iff $A_1 A_2 = A_2 A_1$

Ae^{At}

4) $Ae^{At} = e^{At} A$

$\frac{d}{dt} e^{At}$

5) $\frac{d}{dt} e^{At} = Ae^{At}$

Proof:

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + (At) + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \right)$$
$$= A + A^2 t + \frac{1}{2!} A^3 t^2 + \frac{1}{3!} A^4 t^3 + \cdots$$
$$= A\left(I + (At) + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \right)$$
$$= Ae^{At} \quad \square$$

Initial value problem

Initial

Theorem: Theorem.

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$) and $x(0) = x_0$,

then $x(t) = e^{At} x_0$

$$\boxed{\dot{x}(t)} = \frac{d}{dt}(e^{At}x_0) = \underbrace{Ae^{At}x_0}_{x(t)} = \boxed{Ax(t)}$$

Now the problem bogs down to computing e^{At}

Matrix exponential: diagonal case

$$\text{E.g. } A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$e^{At} = e^{At} = I + (At) + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

$$= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} a_1 t & \\ & a_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}(a_1 t)^2 & \\ & \frac{1}{2!}(a_2 t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (a_1 t) + \frac{1}{2!}(a_1 t)^2 + \cdots & \\ & 1 + (a_2 t) + \frac{1}{2!}(a_2 t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{a_1 t} & \\ & e^{a_2 t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix} x_0$$

Matrix exponential: diagonal case

$$\boxed{A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathbb{R}^{n\times n} \text{ diagonal} \quad \Rightarrow e^{At} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix}}$$

$$\text{E.g. } A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(decoupled)}$$

$$= \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \end{bmatrix}$$

$\dot{x}_1 = a_1 x_1$

$\dot{x}_2 = a_2 x_2$

two one-dim. systems

$a_1 > 0$: unstable

$a_1 < 0$: asym. stable

$a_1 = 0$: stable

$$\Rightarrow \begin{cases} x_1(t) = e^{a_1 t} x_1(0) \\ x_2(t) = e^{a_2 t} x_2(0) \end{cases}$$