Initial value problem

Initial

Theorem: Theorem.

Given $\boxed{\dot{x} = Ax}$ $(A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n)$ and $x(0) = x_0$,

then $x(t) = e^{At} x_0$

$$\boxed{\dot{x}(t)} = \frac{d}{dt}(e^{At} x_0) = \underbrace{Ae^{At} x_0}_{x(t)} = \boxed{Ax(t)}$$

Now the problem bogs down to computing e^{At}

Matrix exponential: diagonal case

$$\text{E.g. } A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$e^{At} = e^{At} = I + (At) + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

$$= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} a_1 t & \\ & a_2 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}(a_1 t)^2 & \\ & \frac{1}{2!}(a_2 t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (a_1 t) + \frac{1}{2!}(a_1 t)^2 + \cdots & \\ & 1 + (a_2 t) + \frac{1}{2!}(a_2 t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{a_1 t} & \\ & e^{a_2 t} \end{bmatrix}$$

$x(t) = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix} x_0$

Matrix exponential: diagonal case

$A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in \mathbb{R}^{n \times n}$ diagonal

$\Rightarrow e^{At} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix}$

$$\text{E.g. } A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \end{bmatrix}$$

(decoupled)

$\dot{x}_1 = a_1 x_1$

$\dot{x}_2 = a_2 x_2$

two one-dim. systems

$\Rightarrow \begin{cases} x_1(t) = e^{a_1 t} x_1(0) \\ x_2(t) = e^{a_2 t} x_2(0) \end{cases}$

$a_1 > 0$: unstable

$a_1 < 0$: asym. stable

$a_1 = 0$: stable

Matrix exponential: diagonalizable case

E.g. $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

Idea: diagonalize A to $A_J = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

where λ_1, λ_2 are eigenvalues of A

and we know how to compute $e^{A_J t} = \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix}$

Find $\quad$ similarity transformation (相似变换)

Find V s.t. $V^{-1}AV = A_J$ (or $A = VA_JV^{-1}$)

A, A_J are similar

$\boxed{Av=\lambda v}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\lambda_1+\lambda_2 = a_{11}+a_{22}$ (trace)

$\lambda_1\lambda_2 = a_{11}a_{22} - a_{12}a_{21}$

Matrix exponential: diagonalizable case

A $\quad A\begin{bmatrix}1\\1\end{bmatrix} = 0 = 0\cdot\begin{bmatrix}1\\1\end{bmatrix}$

E.g. $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

$\boxed{|\lambda I - A| = 0}$

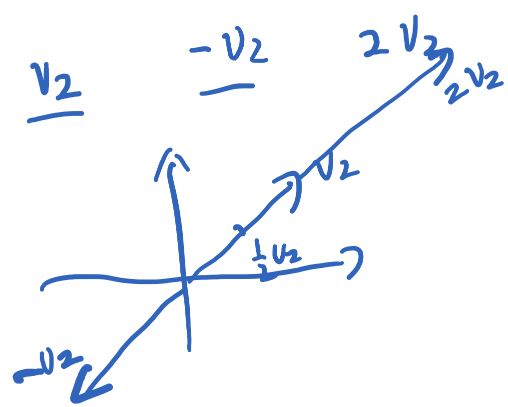
$\begin{vmatrix} \lambda+1 & -1 \\ -2 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2) - 2 = 0$

$\Rightarrow \lambda^2 + 3\lambda = 0$

$\Rightarrow \lambda(\lambda+3) = 0$

First find eigenvalues of A: $\lambda_1 = 0$, $\lambda_2 = -3$

and the corresponding eigenvectors: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}-1\\2\end{bmatrix}$



Matrix exponential: diagonalizable case

E.g. $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

Eigenvalue-eigenvector equation:

$Av_1 =$

$$Av_1 = \lambda_1 v_1$$

$Av_2 =$

$$Av_2 = \lambda_2 v_2$$

$$\Rightarrow A[v_1 \ v_2] = [\lambda_1 v_1 \ \lambda_2 v_2]$$

$$\Rightarrow A[v_1 \ v_2] = \underbrace{[v_1 \ v_2]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{A_J}$$

$$\Rightarrow AV = VA_J$$

$V^{-1}AV = A_J$ or $A = VA_JV^{-1}$

Matrix exponential: diagonalizable case

$AV=$

$$AV = VA_J$$

$A^2V = AAV$

$$A^2V = \underline{AV}A_J = VA_J^2$$

$$\vdots$$

$A^kV=$

$$A^kV = VA_J^k \qquad (k \geq 1)$$

$$e^{At}V = \underline{\underline{e^{At}V}} = (I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots)V$$

$$= V + AVt + \frac{1}{2!}A^2Vt^2 + \frac{1}{3!}A^3Vt^3 + \cdots$$

$$= V + VA_Jt + \frac{1}{2!}VA_J^2t^2 + \frac{1}{3!}VA_J^3t^3 + \cdots$$

$$= V(I + A_Jt + \frac{1}{2!}A_J^2t^2 + \frac{1}{3!}A_J^3t^3 + \cdots)$$

$$= \underline{\underline{Ve^{A_Jt}}}$$

$$\boxed{e^{At} = Ve^{A_Jt}V^{-1}}$$

Matrix exponential: diagonalizable case

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\boxed{\dot{x} = Ax}$$

$$x(t) = e^{At} x_0$$

$$e^{At} V = V e^{A_J t}$$

$$\Rightarrow e^{At} = V e^{A_J t} V^{-1}$$

$$= [v_1 \; v_2] \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} [v_1 \; v_2]^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \cdots$$

Recap

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$),

and suppose A is diagonalizable.

Step 1

Step 1: find eigenvalues of A: $\lambda_1, \ldots, \lambda_n$

and the corresponding eigenvectors $v_1, \ldots, v_n$

Step 2

Step 2: let $A_J := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, and $V := [v_1 \cdots v_n]$

compute $\boxed{e^{At} = Ve^{A_J t}V^{-1}}$

Matrix exponential: non-diagonalizable case

Not

Not all matrices are diagonalizable

Facts:

A

1) $A (\in \mathbb{R}^{n\times n})$ is diagonalizable iff A has n linearly independent eigenvectors

2) If A has n distinct eigenvalues, then A is diagonalizable

Matrix exponential: non-diagonalizable case

upper-triangular

E.g. $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda+1 & -1 \\ 0 & \lambda+1 \end{bmatrix}$

$|\lambda I - A| = (\lambda+1)^2 = 0$

$\boxed{Av = \lambda v}$

Left: $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

R: $-1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = -1$

But there is only one (independent) eigenvector: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (why)

So A is not diagonalizable

Matrix exponential: non-diagonalizable case

Let $M \in \mathbb{R}^{n \times n}$ be a matrix

The null space of M: $\text{null}(M) = \{v \in \mathbb{R}^n \mid Mv = 0\}$

image

The image space of M: $\text{image}(M) = \{v \in \mathbb{R}^n \mid (\exists v')v = Mv'\}$

Fact: $\boxed{\dim(\text{null}(M)) + \dim(\text{image}(M)) = n}$

dim

So $\cancel{\dim}(\text{null}(M)) = n - \dim(\text{image}(M))$

$$= n - \text{rank}(M)$$

Matrix exponential: non-diagonalizable case

E.g. $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

Eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = -1$

But there is only one (independent) eigenvector: $v_1 =$ (why)

Eigenspace

~~Eigenspace~~ of eigenvalue -1: $\text{null}(A - (-1)I) = \text{null}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$

dim

$\dim(\text{null}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})) = n - \text{rank}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = 2 - 1 = 1$

Matrix exponential: non-diagonalizable case

$$\text{E.g. } A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$e^{At} =$

$$e^{At} = e^{\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)t}$$

Matrix exponential: non-diagonalizable case

$$e^{At} \overset{e^{At} =}{=} e^{\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)t}$$

$$e^{(A_1+A_2)t} = e^{A_1 t} e^{A_2 t}$$
iff $A_1 A_2 = A_2 A_1$

$$= e^{\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}t} e^{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}t}$$

(why)

$$= \begin{bmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{bmatrix} \cdot \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}t \right) + \cancel{\frac{1}{2!}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}^2 t^2 + \cdots} \Big)$$

$$= \begin{bmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{bmatrix} \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$$

Matrix exponential: non-diagonalizable case

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is called a nilpotent matrix:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0$$

A is nilpotent if $(\exists k \geq 1)\ A^k = 0$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

is called a Jordan block for -1

Matrix exponential: non-diagonalizable case

In general a Jordan block for λ is

$$A_\lambda \overset{A\lambda}{=} \begin{bmatrix} \lambda & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & \lambda \end{bmatrix}, \text{ where } * \text{ is 0 or 1}$$

$$= \begin{bmatrix} \lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix} + \begin{bmatrix} 0 & * & & \\ & \ddots & \ddots & \\ & & \ddots & * \\ & & & 0 \end{bmatrix}$$

(λI) $\quad$ (A_{nil}: nilpotent matrix)

Matrix exponential: non-diagonalizable case

If $\cancel{A_\lambda} \overset{A_\lambda}{=} \begin{bmatrix} \lambda & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & \lambda \end{bmatrix} \in \mathbb{R}^{m \times m}$

then $e^{A_\lambda t} = \begin{bmatrix} e^{\lambda t} & & \\ & \ddots & \\ & & e^{\lambda t} \end{bmatrix}$

If $\lambda < 0$

If $\lambda = 0$

If $\lambda > 0$

Matrix exponential: non-diagonalizable case

If $A_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{R}^{m\times m}$

then $e^{A_\lambda t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \cdot e^{\lambda t}$

If $\lambda = 0$

If $\lambda > 0$

If $\lambda < 0$

Matrix exponential: non-diagonalizable case

Fact: Every matrix $A \in \mathbb{R}^{n\times n}$ has a Jordan canonical form:

Similar

$$A_J = \begin{bmatrix} A_{\lambda_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & A_{\lambda_p} \end{bmatrix}$$

where $A_{\lambda_i} = \begin{bmatrix} \lambda_i & * & & \\ & \ddots & \ddots & \\ & & \ddots & * \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_i \times m_i}$ is a Jordan block

for eigenvalue λ_i with multiplicity m_i

$$(\det(sI - A) = (s-\lambda_1)^{m_1}\cdots(s-\lambda_p)^{m_p},\ m_1 + \cdots + m_p = n)$$