Matrix exponential: non-diagonalizable case

Fact: Every matrix $A \in \mathbb{R}^{n\times n}$ has a Jordan canonical form:

there exists V s.t. $V^{-1}AV = A_J$ (A, A_J are similar)

$$A_J = \begin{bmatrix} A_{\lambda_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & A_{\lambda_p} \end{bmatrix}$$

where $A_{\lambda_i} = \begin{bmatrix} \lambda_i & * & & \\ & \ddots & \ddots & \\ & & \ddots & * \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_i \times m_i}$ is a Jordan block

for eigenvalue λ_i with multiplicity m_i

$(\det(sI - A) = (s-\lambda_1)^{m_1}\cdots(s-\lambda_p)^{m_p},\ m_1 + \cdots + m_p = n)$

$A_{11} = [0]$

Matrix exponential: non-diagonalizable case

$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

E.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ $\quad A_J = ?$

Eigenvalues of A: $\lambda_1 = \lambda_1 = \quad , \lambda_2 =$

Eigenspace of λ_1 has 1 dimension (why)

So $A_J = \begin{bmatrix} A_{\lambda_1} & 0 \\ 0 & A_{\lambda_2} \end{bmatrix}$ where $A_{\lambda_1} = \qquad , A_{\lambda_2} =$

Then $e^{A_J t} =$

Matrix exponential: non-diagonalizable case

E.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$

For eigenvalue λ_1, there is only one eigenvector $v_1 =$

Define the generalized eigenvector v_1' by $(A - \lambda_1 I)v_1' = v_1$

So $v_1' =$

In general a generalized eigenvector v'
for eigenvalue λ is s.t. $(\exists k)(A - \lambda I)^k v' = 0$

Matrix exponential: non-diagonalizable case

E.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$

For eigenvalue λ_2, the eigenvector is $v_2 =$

Let $V = [v_1 \ v_1' \ v_2]$

Then $AV = A_J V$

Matrix exponential: non-diagonalizable case

$$AV = VA_J$$

$$A^2V = AVA_J = VA_J^2$$

$$\vdots$$

$$A^kV = VA_J^k$$

$$e^{At}V = Ve^{A_J t}$$

So $e^{At} = Ve^{A_J t}V^{-1} =$

Recap

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$),

Step 1: find eigenvalues of A: $\lambda_1, \ldots, \lambda_p$
(with multiplicity $m_1, \ldots, m_p$)
and the (generalized) eigenvectors $v_1, \ldots, v_n$

Step 2: let $A_J = \begin{bmatrix} A_{\lambda_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & A_{\lambda_p} \end{bmatrix}$ be Jordan form where

$$A_{\lambda_i} = \begin{bmatrix} \lambda_i & * & & \\ & \ddots & \ddots & \\ & & \ddots & * \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_i \times m_i}, \text{ and } V := [v_1 \cdots v_n]$$

Recap

In A_{λ_i}: number of 0s $= $ dimension of eigenspace of λ_i

$$= \dim(\text{null}(\lambda_i I - A))$$

$$= n - \text{rank}(\lambda_i I - A)$$

Finally, compute $e^{At} = V e^{A_J t} V^{-1}$

Asymptotic stability criterion

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$),
the origin $x^* = 0$ is asymptotically stable iff
the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A have $\mathrm{Re}(\lambda_i) < 0$

Call A 'stable' (or 'Hurwitz') if
its eigenvalues $\lambda_1, \ldots, \lambda_n$ have $\mathrm{Re}(\lambda_i) < 0$

Stability criterion

Given $\dot{x} = Ax$ ($A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$),
the origin $x^* = 0$ is stable iff
1) the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A have $\mathrm{Re}(\lambda_i) \leq 0$ &
2) for any eigenvalue with $\mathrm{Re}(\lambda_i) = 0$,
Jordan block A_{λ_i} is diagonal

Call A 'semi-stable' if
its eigenvalues $\lambda_1, \ldots, \lambda_n$ have $\mathrm{Re}(\lambda_i) \leq 0$ &
for any eigenvalue with $\mathrm{Re}(\lambda_i) = 0$,
Jordan block A_{λ_i} is diagonal

Example

E.g. $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

Example

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Example

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$