Initial value problem

Given $\dot{x} = Ax$ and $x(0) = x_0$,

then $x(t) = e^{At}x_0$

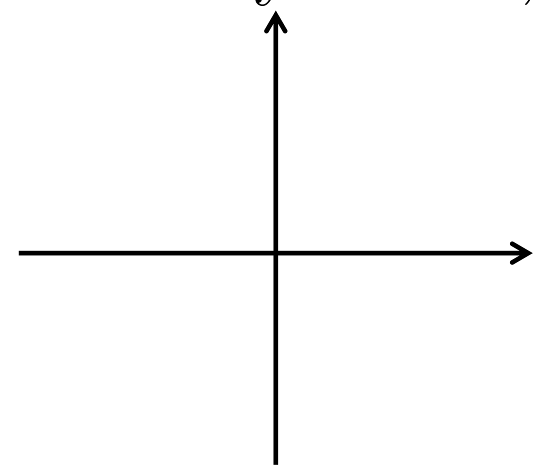
Given $\dot{x} = Ax + Bu$, $y = Cx$, and $x(0) = x_0$,

then $x(t) = ?$, $y(t) = ?$

Controllability

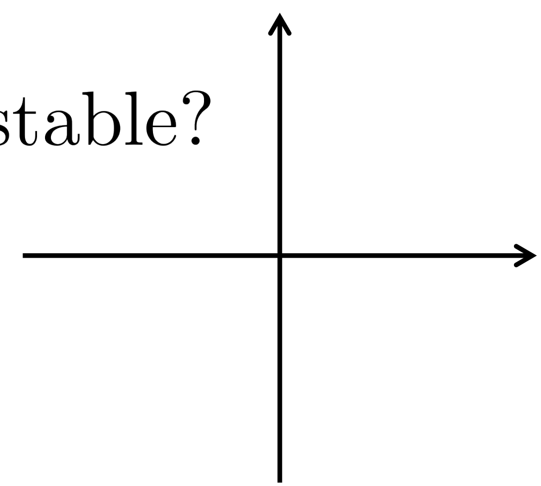
Main question

For $\dot{x} = Ax$ ($A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$) and $x(0) = x_0$,
if the origin $x^* = 0$ is already asymptotically stable,
then we are done



If not, consider $\dot{x} = Ax + Bu$

Can we apply control input u to
make $\dot{x} = Ax + Bu$ asymptotically stable?

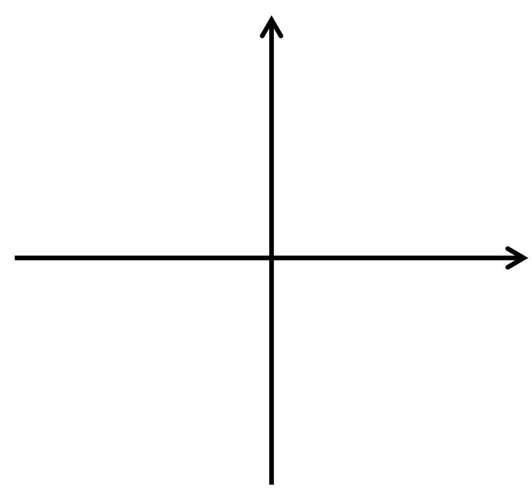


Reachability problem

What states in $\mathbb{R}^n$ can be reached by choice of control?

More formally: given $\dot{x} = Ax + Bu$ and
an arbitrary $v \in \mathbb{R}^n$ and time $t_1 > 0$,

can we find control inputs $u(0), \ldots, u(t_1)$ such that
$x(0) = 0$ and $x(t_1) = v$



First, for $\dot{x} = Ax + Bu$ and $x(0) = 0$,

$x(t_1) =$

So if v belongs to the column span (image) of $[B \; AB \; A^2B \cdots A^{n-1}B \; A^nB \cdots]$, then $u(\tau)$ $(\tau \in [0, t_1])$ can be chosen to render $x(t_1) = v$

Second, we need Cayley-Hamilton Theorem:

If $p(\lambda)$ is the characteristic polynomial of A:
$p(\lambda) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$
then $p(A) = a_n A^n + \cdots + a_1 A + a_0 I = 0$

This means $a_n A^n = -a_{n-1} A^{n-1} - \cdots - a_1 A - a_0 I$
'A^n is a linear combination of $A^{n-1}, \ldots, A, I$'

So by Cayley-Hamilton, it is sufficient that v belongs to the column span (image) of $[B \; AB \cdots A^{n-1} B]$

Third, the reachability problem is solvable iff

$$\operatorname{rank}[B\ AB \cdots A^{n-1}B] = n$$

The solution is: $u(\tau) =$

Controllability

Write $W_c := [B \ AB \cdots A^{n-1}B]$,
and call W_c the controllability matrix

Defn. Consider $\dot{x} = Ax + Bu$.
The pair (A, B) is *controllable* if $\text{rank} W_c = n$

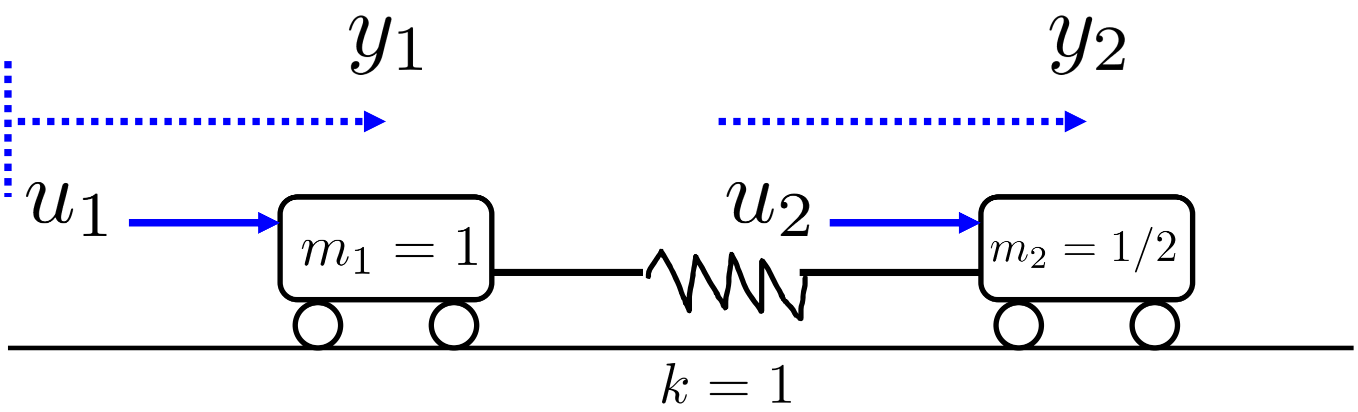
Controllability

Controllability is invariant under coordinate transformation:

Consider $\dot{x} = Ax + Bu$ and a new state $\tilde{x} = V^{-1}x$ for some invertible V (thus $x = V\tilde{x}$)

Then $\dot{\tilde{x}} = V^{-1}\dot{x} =$

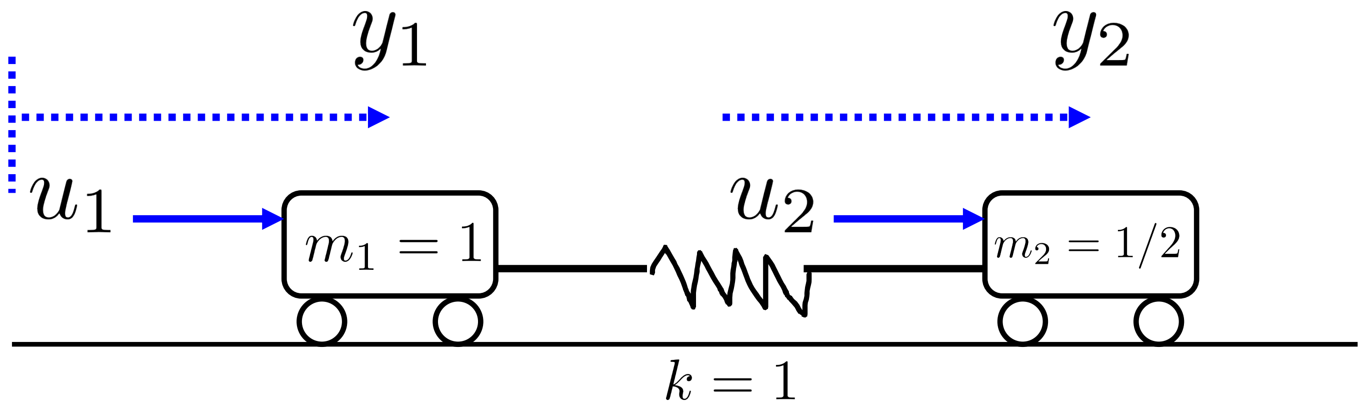
Example: cart-spring



$$m_1 \ddot{y}_1 = u_1 + k(y_2 - y_1)$$
$$m_2 \ddot{y}_2 = u_2 + k(y_1 - y_2)$$

$$\text{State vector: } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$$

Example: cart-spring



Then $\dot{x} = Ax + Bu$, where $A =$ $B =$

$W_c =$

$\text{rank} W_c =$