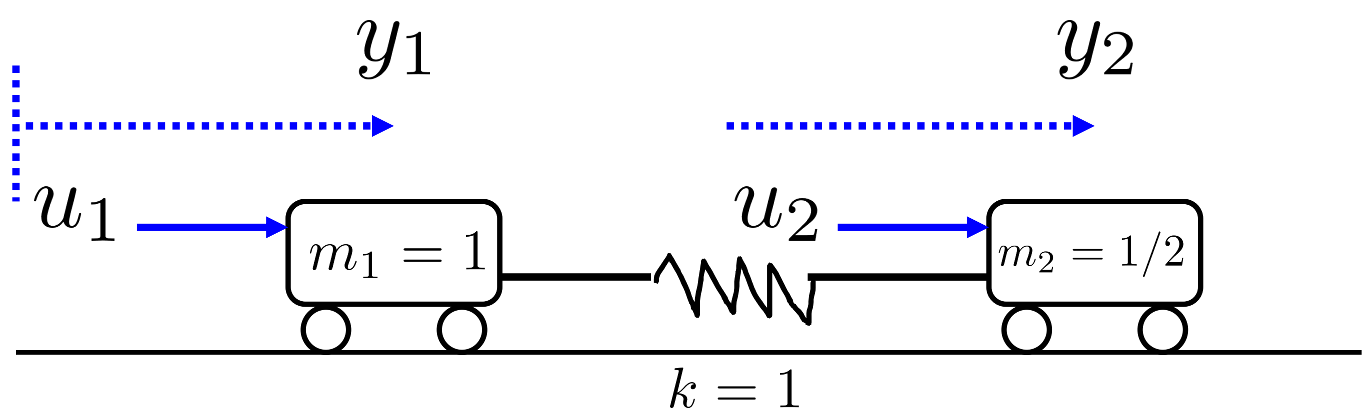
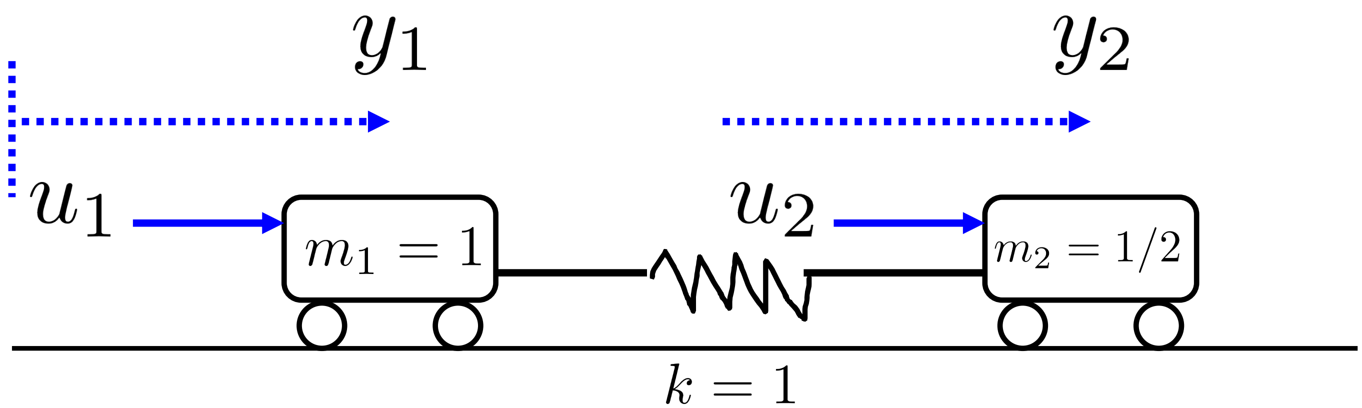
Example: cart-spring



$$m_1 \ddot{y}_1 = u_1 + k(y_2 - y_1)$$
$$m_2 \ddot{y}_2 = u_2 + k(y_1 - y_2)$$

$$\text{State vector: } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$$

Example: cart-spring



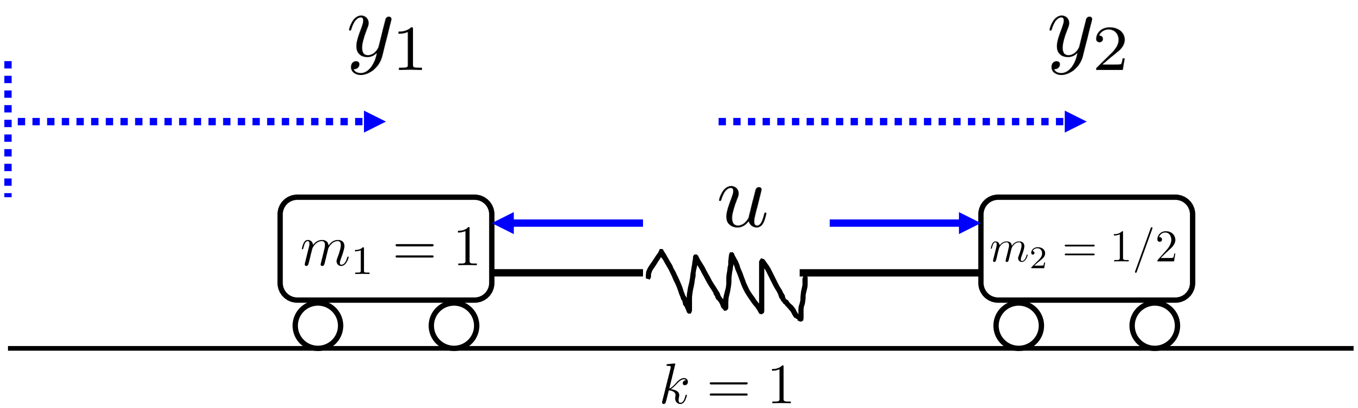
Then $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$

$$W_c = [B \;\; AB \;\; A^2B \;\; A^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & -4 \\ 0 & 2 & 0 & 0 & 2 & -4 & 0 & 0 \end{bmatrix}$$

$$\text{rank} W_c = 4$$

(A, B) controllable

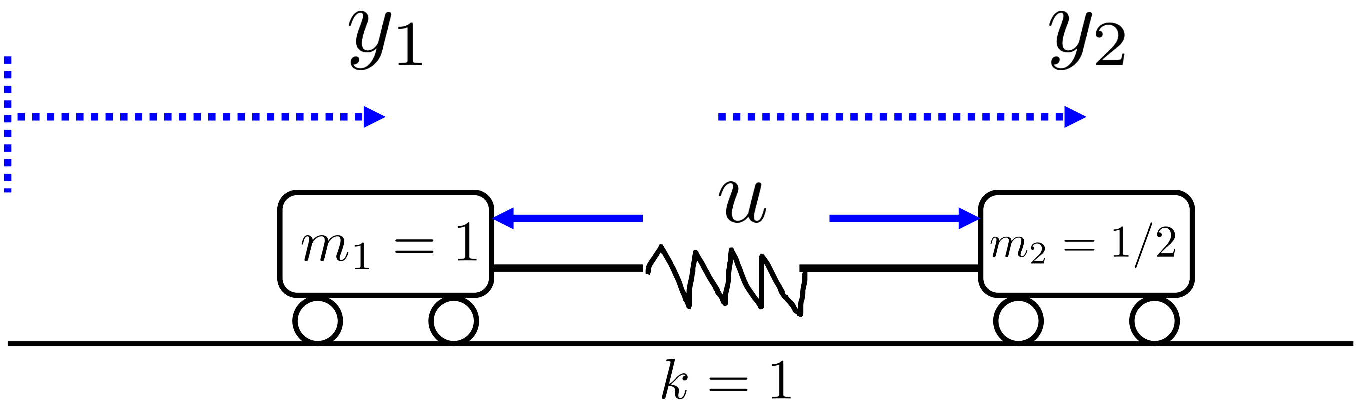
Example: cart-spring



$$m_1 \ddot{y}_1 = -u + k(y_2 - y_1)$$
$$m_2 \ddot{y}_2 = u + k(y_1 - y_2)$$

$$\text{State vector: } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$$

Example: cart-spring



Then $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

$W_c = [B \;\; AB \;\; A^2B \;\; A^3B] = \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -6 \\ 2 & 0 & -6 & 0 \end{bmatrix}$

$\text{rank} W_c = 2 < 4$

(A, B) not controllable

PBH test for controllability

Consider $\dot{x} = Ax + Bu$. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

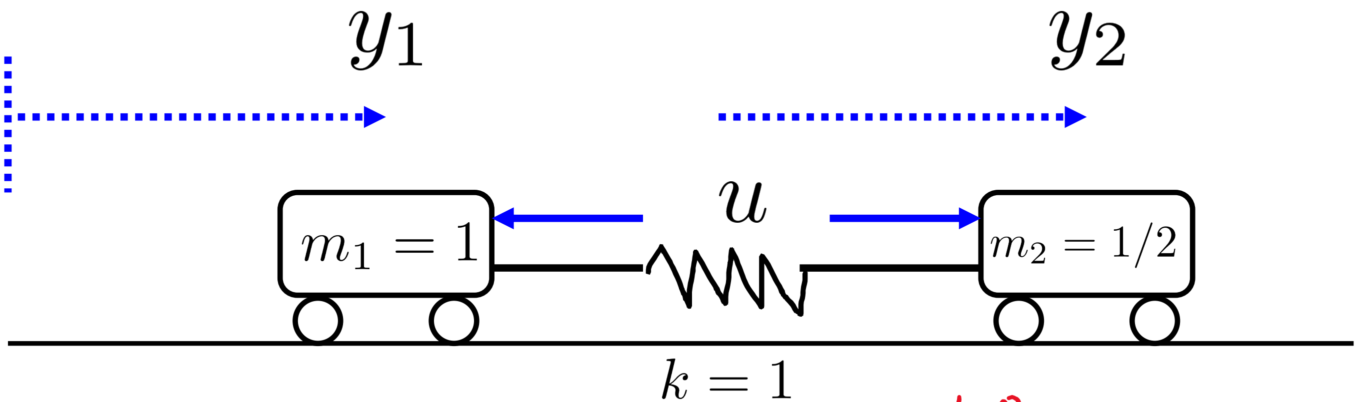
Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A.

Then the pair (A, B) is *controllable* iff

$$(\forall i = 1, \ldots, n) \ \mathrm{rank}\underbrace{[A - \lambda_i I \quad B]}_{n \times (n+m)} = n$$

$$W_c = [\underbrace{B}_{n \times m} \quad \underbrace{AB}_{n \times m} \quad \cdots \quad \underbrace{A^{n-1}B}_{n \times m}]_{n \times (nm)}$$

Example: cart-spring



PBH Test

Then $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

Step 1: compute eigenvalues of A: $0, 0, \sqrt{3}j, -\sqrt{3}j$

Step 2: for eigenvalue $\lambda = 0$

$\mathrm{rank}[A - \lambda I \;\; B] = \mathrm{rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & 0 & 2 \end{bmatrix} = 3 < 4$

$\lambda = \sqrt{3}j, -\sqrt{3}j$, $\mathrm{rank}[A - \lambda I \;\; B] = 4$

PBH test for controllability

Defn. Consider $\dot{x} = Ax + Bu$.
Let λ be an eigenvalue of A.
Say λ is controllable if $\mathrm{rank}[A - \lambda I \quad B] = n$

Controllability with single input

Let's start with the simplest case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$a_1, a_2, a_3 \in \mathbb{R}$

Is (A, B) controllable?

$$W_c = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_3 \\ 1 & -a_3 & -a_2 + a_3^2 \end{bmatrix}$$

$\operatorname{rank} W_c = 3$ $\quad (A, B)$ controllable!

Controllability with single input

Let's start with the simplest case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What is the characteristic polynomial of A?

$\det(\lambda I - A) = \lambda^3 + a_3\lambda^2 + a_2\lambda + a_1$

Controllability with single input

Control Canonical Form (CCF):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

1) (A, B) is controllable

2) characteristic polynomial of A is
$\lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1$

Controllability with single input

Let's start with the simplest case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$

E.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $a_1 = 1, a_2 = -1, a_3 = -1$

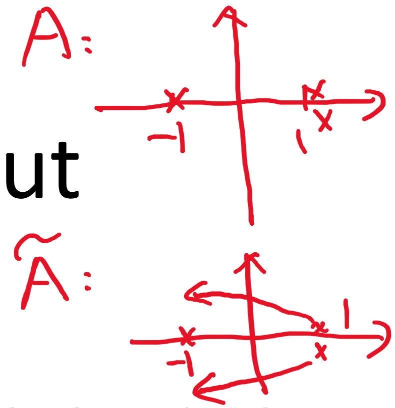
So characteristic polynomial of A: $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\Rightarrow (\lambda - 1)^2 (\lambda + 1) = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 1, \quad \lambda_3 = -1$

Is $\dot{x} = Ax$ stable? X

Controllability with single input



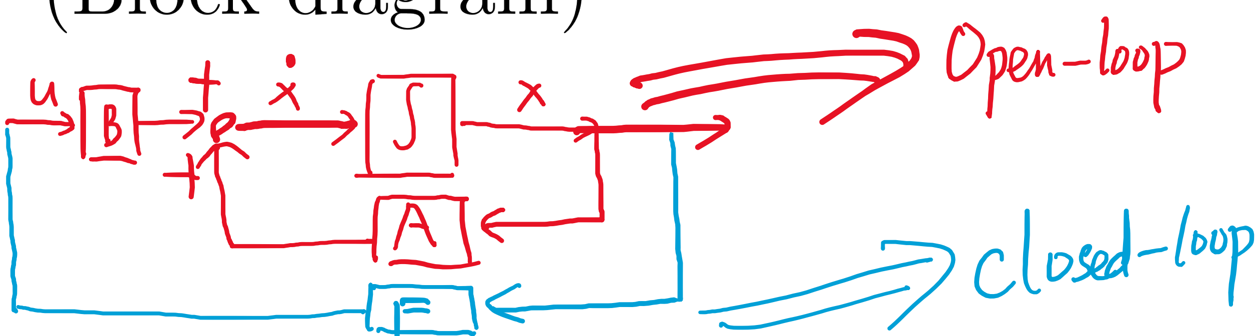
Whenever $\dot{x} = Ax$ is unstable,
can we use a single input u to 'stabilize' the system:
$\dot{x} = Ax + Bu$ is stable?

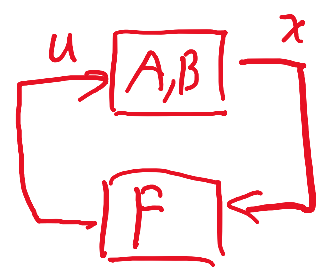
Consider the form of 'state feedback': $\boxed{u = Fx}$

Then $\dot{x} = Ax + Bu = Ax + BFx = \underbrace{(A+BF)}_{\tilde{A}}x$

$\boxed{\dot{x} = \tilde{A}x}$

(Block diagram)





Controllability with single input

Goal: design F such that $A + BF$ is stable

E.g. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $a_1 = 1, a_2 = -1, a_3 = -1$

A eigs: $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$

Desired eigenvalues of $A + BF$: $\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -1$

$u = Fx$

char. equation:

$(\lambda+2)(\lambda+3)(\lambda+1) = 0$

Let $F = [F_1 \;\; F_2 \;\; F_3]$

Then $A + BF =$

$$A+BF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1+F_1 & 1+F_2 & 1+F_3 \end{bmatrix}$$

char. poly. : $\lambda^3 - (1+F_3)\lambda^2 - (1+F_2)\lambda - (-1+F_1)$

But we need: $(\lambda+2)(\lambda+3)(\lambda+1)$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$\begin{cases} -(1+F_3) = 6 \\ -(1+F_2) = 11 \\ -(-1+F_1) = 6 \end{cases} \Rightarrow \begin{cases} F_1 = -5 \\ F_2 = -12 \\ F_3 = -7 \end{cases} \Rightarrow F = \begin{bmatrix} -5 & -12 & -7 \end{bmatrix}$$

$$\boxed{u = Fx}$$

Controllability with single input

Eigenvalue Assignment (or Pole Placement)

Consider $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$
and assume (A, B) is in control canonical form.
Let $u = Fx$ (state feedback), $F = [F_1 \cdots F_n]$ and
desired eigenvalues of $A + BF$ be $\lambda_1, \ldots, \lambda_n$ (stable)

Then F can always be designed and
is computed by comparing the coefficients of

$$\lambda^n + (a_n - F_n)\lambda^{n-1} + \cdots + (a_2 - F_2)\lambda + (a_1 - F_1)$$

and $(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$

Controllability with single input

Consider $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$
Assume (A, B) is controllable
(but may not be in control canonical form)

Let the characteristic polynomial of A be
$\lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1$

Then there exists an invertible matrix W s.t.

$$W^{-1}AW = \tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, W^{-1}B = \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Controllability with single input

'Any controllable (A, B) can be similarly transformed to control canonical form'

This $W = W_c \tilde{W}_c^{-1}$ where
$W_c = [B \ AB \cdots A^{n-1}B]$ and $\tilde{W}_c = [\tilde{B} \ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}]$

(A, B) (controllable) $\longrightarrow$ $(\tilde{A}, \tilde{B})$ (CCF)

$$A + BF = A + \underline{B\tilde{F}W^{-1}}$$
$$= W\tilde{A}W^{-1} + W\tilde{B}\tilde{F}W^{-1}$$
$$= W(\tilde{A} + \tilde{B}\tilde{F})W^{-1}$$

$F = \tilde{F}W^{-1}$ (check if $A + BF$ stable!) $\longleftarrow$ $\tilde{F}$ (s.t. $\tilde{A} + \tilde{B}\tilde{F}$ stable)

Controllability with single input

$$\text{E.g. } A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}, \ B = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

$$W_c = \begin{bmatrix} -3 & -6 & -7 \\ 3 & 5 & 5 \\ 1 & 2 & 2 \end{bmatrix}, \quad \text{rank } W_c = 3, \quad (A,B) \text{ controllable}$$

(A,B) is NOT CCF.

Characteristic polynomial of A:

$$\lambda^3 - \lambda^2 - \lambda + 1$$

Controllability with single input

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$

So the corresponding CCF: $\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus $\tilde{W}_c = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} \end{bmatrix} = \ldots$

$W = W_c \tilde{W}_c^{-1} = \ldots$

Controllability with single input

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$

$W^{-1} =$

Finally check:
$W^{-1}AW = \tilde{A}$
$W^{-1}B = \tilde{B}$

Controllability with single input

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$

Assume the desired eigenvalues: $-1, -2, -3$

For $(\tilde{A}, \tilde{B})$, we already found $\tilde{F} = [-5 \; -12 \; -7]$
s.t. $\tilde{A} + \tilde{B}\tilde{F}$ has the desired eigenvalues

Now set $F = \tilde{F}W^{-1}$
Check $A + BF$ also has the desired eigenvalues

Controllability with single input

$\boxed{\text{Eigenvalue Assignment}}$ (or Pole Placement)

Consider $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$
and assume (A, B) is controllable.
Let $u = Fx$ (state feedback), $F = [F_1 \cdots F_n]$ and
desired eigenvalues of $A + BF$ be $\lambda_1, \ldots, \lambda_n$ (stable)

Then F can be designed by the following 3 steps:

Controllability with single input

Eigenvalue Assignment (or Pole Placement)

Step 1. Compute characteristic polynomical of A:
$\lambda^n + a_n\lambda^{n-1} + \cdots + a_2\lambda + a_1$ $\Rightarrow \tilde{A}, \tilde{B}$ (CCF), $\tilde{W}_c$
and compute $W = W_c\tilde{W}_c^{-1}$

Step 2. Compute $\tilde{F}$ s.t. $\tilde{A} + \tilde{B}\tilde{F}$ have desired eigenvalues

Step 3. Set $F = \tilde{F}W^{-1}$