Controllability with multiple inputs

Let's now turn to the general case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$

Check (A, B) is controllable ✓

$\text{rank}\left(W_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}\right) = 3$

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E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$

Step 1: Let $B_1 = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$ be the first column of B

$\text{rank}[B_1 \ AB_1 \ A^2B_1] = \text{rank}\begin{bmatrix} -3 & -9 & -3 \\ 0 & 6 & 0 \\ 0 & 3 & 0 \end{bmatrix} = 2 < 3$

So (A, B_1) is not controllable

Controllability with multiple inputs

Let's now turn to the general case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$

Step2:

But for $\hat{F} = [1\ 2\ 3]$, $(A + B\hat{F}, B_1)$ is controllable:
$\operatorname{rank}[B_1\ (A + B\hat{F})B_1\ (A + B\hat{F})^2 B_1] = \operatorname{rank}\begin{bmatrix} -3 & 0 & 33 \\ 0 & -3 & -33 \\ 0 & 0 & -12 \end{bmatrix} = 3$

Step3:

So for $(A + B\hat{F}, B_1)$ (single input)
design $\tilde{F}$ to assign eigenvalues of $(A + B\hat{F}) + B_1\tilde{F}$

Controllability with multiple inputs

Let's now turn to the general case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

E.g. $A = \begin{bmatrix} 3 & -5 & 18 \\ -2 & 5 & -16 \\ -1 & 2 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$

step 4:

Finally set $F := \hat{F} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{F}$

Then $A + BF = A + B\left(\hat{F} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{F}\right)$

$= A + B\hat{F} + B\begin{bmatrix} 1 \\ 0 \end{bmatrix}\tilde{F} = A + B\hat{F} + B_1\tilde{F}$

Controllability with multiple inputs

Let's now turn to the general case:
$\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

Fact: Let (A, B) be controllable,
and B_1 be the first column of B (and $B_1 \neq 0$).
Then there always exists $\hat{F}$ s.t.
$(A + B\hat{F}, B_1)$ is controllable.

Controllability with multiple inputs

Eigenvalue Assignment (or Pole Placement)

Consider $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$
and let $u = Fx$ (state feedback).
Then all the eigenvalues of $A + BF$ are freely assignable
$\boxed{\text{iff}}$ (A, B) is controllable

1967 W. Murray Wonham first proved this result.

Stabilizability

To stabilize an unstable $\dot{x} = Ax + Bu$ by state feedback, all we need to do is to move the unstable eigenvalues of A

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues of A: $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$

So $\dot{x} = Ax + Bu$ is unstable

To stabilize $\dot{x} = Ax + Bu$, we only need to move the unstable λ_1, λ_2 to the (open) left half plane

Stabilizability

Consider $\dot{x} = Ax + Bu$ and $u = Fx$

Say (A, B) is *stabilizable* if $(\exists F) A + BF$ is stable

Note: if (A, B) is controllable, then (A, B) is *stabilizable*

Proposition: Consider $\dot{x} = Ax + Bu$ and
let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A.
Then (A, B) is stabilizable iff

$$(\forall i = 1, \ldots, n) \mathrm{Re}(\lambda_i) \geq 0 \Rightarrow \mathrm{rank}[A - \lambda_i I \quad B] = n$$

Stabilizability

Note: (A, B) is *stabilizable* $\not\Rightarrow$ (A, B) is controllable

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues of A: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$

$$\operatorname{rank}[A - \lambda_1 I \quad B] = \operatorname{rank}\begin{bmatrix} -1 & 1 & 1 & 0 \\ -2 & 2 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 3$$

$$\operatorname{rank}[A - \lambda_2 I \quad B] = 3$$

stabilizable

$$\operatorname{rank}[A - \lambda_3 I \quad B] = \operatorname{rank}\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 < 3$$

NOT controllable

Kalman decomposition

$\dot{x} = Ax + Bu$ can be decomposed into a controllable part and an uncontrollable part

$$
\begin{array}{ccc}
 & A_{11} \quad A_{12} & \quad B_1 \\
\text{E.g. } A = & \left[\begin{array}{cc|c} 0 & 1 & 1 \\ -2 & 3 & 0 \\ \hline 0 & 0 & -1 \end{array}\right], \; B = & \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] \\
 & A_{21} \quad A_{22} & \quad B_2
\end{array}
$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

Controllable part: $\dot{\tilde{x}}_1 = A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + B_1 u$

(A_{11}, B_1) controllable

Uncontrollable part: $\dot{\tilde{x}}_2 = A_{22}\tilde{x}_2$

no control input u

(A, B) is stabilizable because $A_{22} = -1$ is stable

Kalman decomposition

$\dot{x} = Ax + Bu$ can be decomposed into a controllable part and an uncontrollable part

A_{11} $\quad$ B_1

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Design F_1 s.t. $A_{11} + B_1 F_1$ is stable $\quad -1, -2$

$(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2 \quad \cdots \text{①}$

$A_{11} + [F_{11} \;\; F_{12}] B_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + [F_{11} \;\; F_{12}] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ F_{11} & F_{12} \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -2+F_{11} & 3+F_{12} \end{bmatrix}$

$\boxed{F_1 = [0 \;\; -6]}$

$\lambda^2 - (3+F_{12})\lambda - (-2+F_{11}) \quad \cdots \text{②}$

Kalman decomposition

$\dot{x} = Ax + Bu$ can be decomposed into a controllable part and an uncontrollable part

$$\text{E.g. } A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Design F_1 s.t. $A_{11} + B_1 F_1$ is stable

Set $F = [F_1 \;\; 0]$

$$A + BF = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 F_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Kalman decomposition

E.g. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

$$\text{rank} W_c = \text{rank} \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -6 \\ 2 & 0 & -6 & 0 \end{bmatrix} = 2 < 4$$

The image space (column span) of $\overset{W_c}{\cancel{M}}$:

$\text{image}(\overset{W_c}{\cancel{M}}) = \{ v \in \mathbb{R}^n \mid (\exists v') v = \overset{W_c}{\cancel{M}} v' \} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

The set of all reachable states is

the 2-dim subspace spanned by these two columns

Let $e_1 := \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, $e_2 := \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

Add 2 more (independent) vectors to form a basis for $\mathbb{R}^4$

(say standard basis) $e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now write $V := [e_1 \ e_2 \ e_3 \ e_4]$

V is invertible: $V^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

Define a new state $\tilde{x} := V^{-1}x$ (thus $x = V\tilde{x}$)

$$
\begin{aligned}
\text{So } \dot{\tilde{x}} = V^{-1}\dot{x} &= V^{-1}(Ax + Bu) \\
&= V^{-1}AV\tilde{x} + V^{-1}Bu \\
&=: \tilde{A}\tilde{x} + \tilde{B}u \\
&= \begin{array}{cc} A_{11} & A_{12} \\ \multicolumn{2}{c}{\left[\begin{array}{cc|cc} 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \\ A_{21} & A_{22} \end{array} \tilde{x} + \begin{array}{c} B_1 \\ \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \\ B_2 \end{array} u
\end{aligned}
$$

Define a new state $\tilde{x} := V^{-1}x$ (thus $x = V\tilde{x}$)

$$
\begin{aligned}
\text{So } \dot{\tilde{x}} = V^{-1}\dot{x} &= V^{-1}(Ax + Bu) \\
&= V^{-1}AV\tilde{x} + V^{-1}Bu \\
&=: \tilde{A}\tilde{x} + \tilde{B}u
\end{aligned}
$$

A_{11} A_{12} B_1

$$
= \left[\begin{array}{cc|cc} 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tilde{x} + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] u
$$

0 A_{22} 0

Correspondingly $\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$ $\tilde{x}_1 \in \mathbb{R}^2$, $\tilde{x}_2 \in \mathbb{R}^2$

Controllable part: $\dot{\tilde{x}}_1 = A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 + B_1 u$

Uncontrollable part: $\dot{\tilde{x}}_2 = A_{22}\tilde{x}_2$

Kalman decomposition

In general $\dot{x} = Ax + Bu$ can be decomposed into a controllable part and an uncontrollable part:

Step 1: Let $\{e_1, \ldots, e_k\}$ be a basis for image(W_c). Complement it to form a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$

Step 2: Let $V := [e_1 \cdots e_n]$ and define $\tilde{x} := V^{-1}x$

Then $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$, where $\tilde{A} = V^{-1}AV$, $\tilde{B} = V^{-1}B$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

(A_{11}, B_1) is controllable

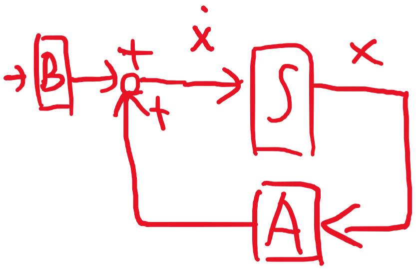
Kalman decomposition

Block diagram:

Controllable subsystem

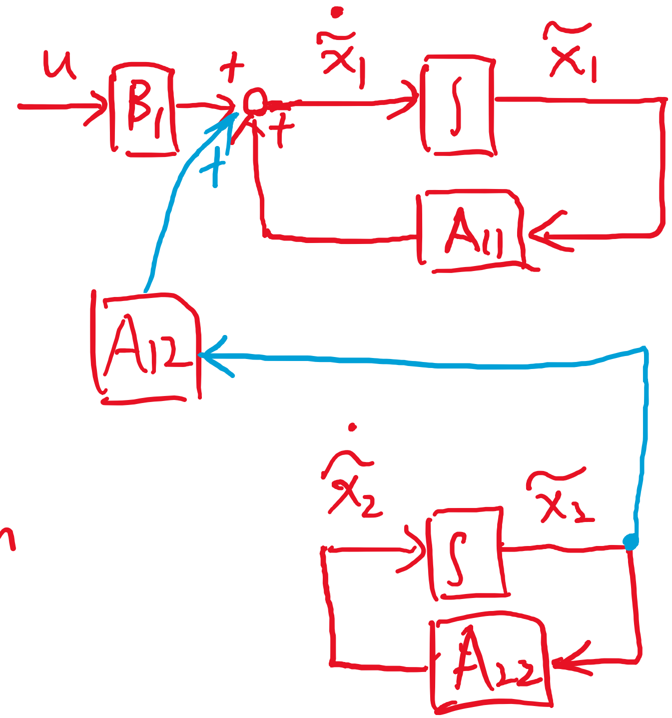
$$\dot{\tilde{x}}_1 = \underline{A_{11}\tilde{x}_1 + B_1 u} + \boxed{A_{12}\tilde{x}_2}$$

$$\dot{x} = Ax + Bu$$



V

similarity transformation



$$\dot{\tilde{x}}_2 = A_{22}\tilde{x}_2$$

Uncontrollable subsystem

Stabilizability

Proposition:
Consider $\dot{x} = Ax + Bu$ and
the decomposed system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

Then (A, B) is *stabilizable* iff A_{22} is stable

Stabilizability

Consider $\dot{x} = Ax + Bu$ and assume (A, B) is *stabilizable*

Design $u = Fx$ to stabilize the system:

Step 1: Decompose the system into $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

Step 2: Compute F_1 s.t. $A_{11} + B_1 F_1$ is stable

Step 3: Set $\tilde{F} = [F_1 \;\; 0]$. So $\tilde{A} + \tilde{B}\tilde{F} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Step 4: Set $F = \tilde{F} V^{-1}$

$$A + BF = A + B\tilde{F}V^{-1} = V\tilde{A}V^{-1} + V\tilde{B}\tilde{F}V^{-1}$$
$$= V(\tilde{A} + \tilde{B}\tilde{F})V^{-1}$$