Observability

Main question

For $\dot{x} = Ax + Bu$ with (A, B) stabilizable
we know how to design $u = Fx$ to stabilize the system

What if we cannot observe all the state variables?

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For $\dot{x} = Ax + Bu$ with (A, B) stabilizable
we know how to design $u = Fx$ to stabilize the system

What if we cannot observe all the state variables?

Consider $\dot{x} = Ax + Bu$, $y = Cx$

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

All we can observe is y

$y = Cx = x_1$

Main question

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we know how to design $u = Fx$ to stabilize the system

What if we cannot observe all the state variables?

Consider $\dot{x} = Ax + Bu$, $y = Cx$

All we can observe is y

$\boxed{u = Fy\ ?}$

Can we design control input u based on output y to make $\dot{x} = Ax + Bu$ asymptotically stable?

(Output feedback)

State reconstruction problem

What states in $\mathbb{R}^n$ can be reconstructed by observing y?

Idea: Observe y $\longrightarrow$ reconstruct x $\longrightarrow$ $u = Fx$?

State reconstruction problem

What states in $\mathbb{R}^n$ can be reconstructed by observing y?

More formally: given $\dot{x} = Ax$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
time $t_1 > 0$ and observe $y(0), \ldots, y(t_1)$,
can we find the initial state $x(0)$?

(If we can find $x(0)$, we can compute $x(t)$, $t > 0$)

$$\boxed{x(t) = e^{At} x(0)}$$

$y(t_1) = C\,x(t_1)$

$$y(t_1) = Ce^{At_1}x(0)$$

$$= C\left(I + At_1 + \frac{1}{2!}A^2t_1^2 + \frac{1}{3!}A^3t_1^3 + \cdots\right)x(0)$$

$$= \underbrace{\left(C + CAt_1 + CA^2\frac{t_1^2}{2!} + CA^3\frac{t_1^3}{3!} + \cdots\right)}_{M}x(0)$$

$$y(t_1) = Ce^{At_1}x(0)$$
$$=$$

$$\boxed{y(t_1) = Mx(0)}$$

$\boxed{y = Mx}$

So $x(0)$ can be uniquely determined iff the null space (kernel) of M is zero:
$\text{null}(M) = \{x \mid Mx = 0\} = 0$

$$M = C + CA + CA^2\frac{t_1^2}{2!} + CA^3\frac{t_1^3}{3!} + \cdots$$

Claim: $\operatorname{null}(M) = 0 \iff \operatorname{null}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \\ \vdots \end{bmatrix}\right) = 0$

Proof: ($\Rightarrow$) Suppose $\operatorname{null}(M) = 0$.

Want to prove $\operatorname{null}\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} = 0$.

Let $\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} x = 0$. Want to prove $x = 0$.

Then $Cx = 0$, $CAx = 0$, $CA^2x = 0$, $\ldots$

So $Mx = Cx + CAx + CA^2x \cdot \frac{t_1^2}{2!} + \cdots = 0$

Since $\operatorname{null}(M) = 0$, we have $x = 0$.

We conclude $\operatorname{null}\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} = 0$.

($\Leftarrow$) $\ldots$

$$\text{Fact: } \operatorname{null}(M) = 0 \Leftrightarrow \operatorname{null}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \\ \vdots \end{bmatrix}\right) = 0$$

$$\Leftrightarrow \operatorname{image}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^n \\ \vdots \end{bmatrix}\right) = \mathbb{R}^n$$

null(M) + image(M)
$= \mathbb{R}^n$

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$$\boxed{\text{Cayley-Hamilton}}$$

$$\Leftrightarrow \mathrm{image}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = \mathbb{R}^n$$

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$$\Leftrightarrow \operatorname{image}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = \mathbb{R}^n \quad \Leftrightarrow \operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

Conclusion: the state reconstruction problem is solvable iff

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$$

pn rows
n columns

~~pn x n~~

$\dot{x} = Ax$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$

$y = Cx$ $y \in \mathbb{R}^p$, $C \in \mathbb{R}^{p \times n}$

Observability

Write $W_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$,

and call W_o the observability matrix

Observability

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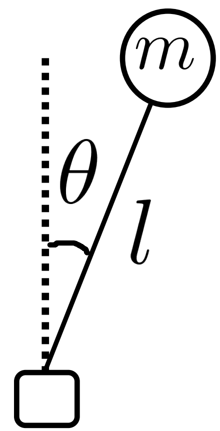
$W_c = [B \ AB \ \cdots \ A^{n-1}B]$
controllability matrix

and call W_o the observability matrix

(A,B) controllable
if $\operatorname{rank} W_c = n$

Defn. Consider $\dot{x} = Ax$, $y = Cx$.
The pair (C, A) is *observable* if $\operatorname{rank} W_o = n$

Example: inverted pendulum



$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}$$

rank $W_o = 2$

(C, A) observable

Observability

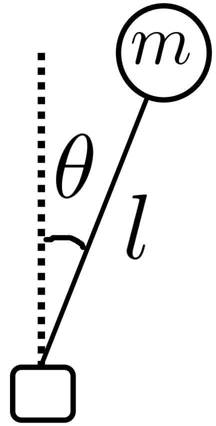
$$\text{Observability matrix } W_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ for } (C, A)$$

Note: $W_o^T = [C^T \;\; A^T C^T \cdots (A^T)^{n-1} C^T]$ is *controllability matrix* for (A^T, C^T)

$\boxed{\dot{x} = A^T x + C^T u}$

So (C, A) is observable iff (A^T, C^T) is controllable (why?)

Example: inverted pendulum



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$$A^T = \begin{bmatrix} 0 & \frac{g}{l} \\ 1 & 0 \end{bmatrix}, \; C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$W_c = [C^T \quad A^T C^T] = \begin{bmatrix} 0 & \frac{g}{l} \\ 1 & 0 \end{bmatrix},$$

rank $W_c = 2$, (A^T, C^T) controllable

Observability dual with controllability

$\dot{x} = Ax,\ y = Cx$ (observability) (controllability)

(C, A) is *observable* (A^T, C^T) is *controllable*

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

$$\operatorname{rank}[C^T \ \ A^T C^T \cdots (A^T)^{n-1} C^T] = n$$

Observability dual with controllability

$$\dot{x} = Ax,\ y = Cx \text{ (observability)}$$

(controllability)

(C, A) is *observable*

(A^T, C^T) is *controllable*

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$$\text{rank}[C^T \;\; A^T C^T \cdots (A^T)^{n-1} C^T] = n$$

PBH Test

$$(\forall i = 1, \ldots, n) \text{ rank}\begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n$$

$(\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A)$

$$(\forall i = 1, \ldots, n) \text{ rank}[A^T - \lambda_i I \;\; C^T] = n$$

$(\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^T)$

Observability dual with controllability

$\dot{x} = Ax,\ y = Cx$ (observability)

(controllability)

(C, A) is *observable*

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$(\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A)$

$$(\forall i = 1, \ldots, n)\ \text{rank}[A^T - \lambda_i I \ \ C^T] = n$$

$(\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^T)$

$(\exists L)$eigenvalues of $A + LC$ are freely assignable

$(\exists L^T)$eigenvalues of $A^T + C^T L^T$ are freely assignable

$(A+BF)$

Observability dual with controllability

$\dot{x} = Ax,\ y = Cx$ (observability) (controllability)

observable canonical form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

control canonical form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Observability dual with controllability

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(C, A) is *detectable*:
$(\exists L) A + LC$ is stable

(A^T, C^T) is *stabilizable*:
$(\exists L^T) A^T + C^T L^T$ is stable

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(A^T, C^T) is *stabilizable*:
$(\exists L^T) A^T + C^T L^T$ is stable

$(\forall i = 1, \ldots, n)\ \mathrm{Re}(\lambda_i) \geq 0 \Rightarrow$
$\mathrm{rank} \begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n$

PBH

$(\forall i = 1, \ldots, n)\ \mathrm{Re}(\lambda_i) \geq 0 \Rightarrow$
$\mathrm{rank} [A^T - \lambda_i I \quad C^T] = n$

Kalman decomposition

$\dot{x} = Ax + Bu$ can be decomposed into a controllable part and an uncontrollable part:

Kalman decomposition

$\dot{x} = Ax + Bu$, $y = Cx$ can be decomposed into a controllable part and an uncontrollable part:

There exists an invertible V_c, with $\tilde{x} := V_c^{-1}x$

s.t. $\dot{\tilde{x}} = V_c^{-1}AV_c\tilde{x} + V_c^{-1}Bu$, $y = CV_c\tilde{x}$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ \boxed{0} \end{bmatrix} u$$

controllable

uncontrollable

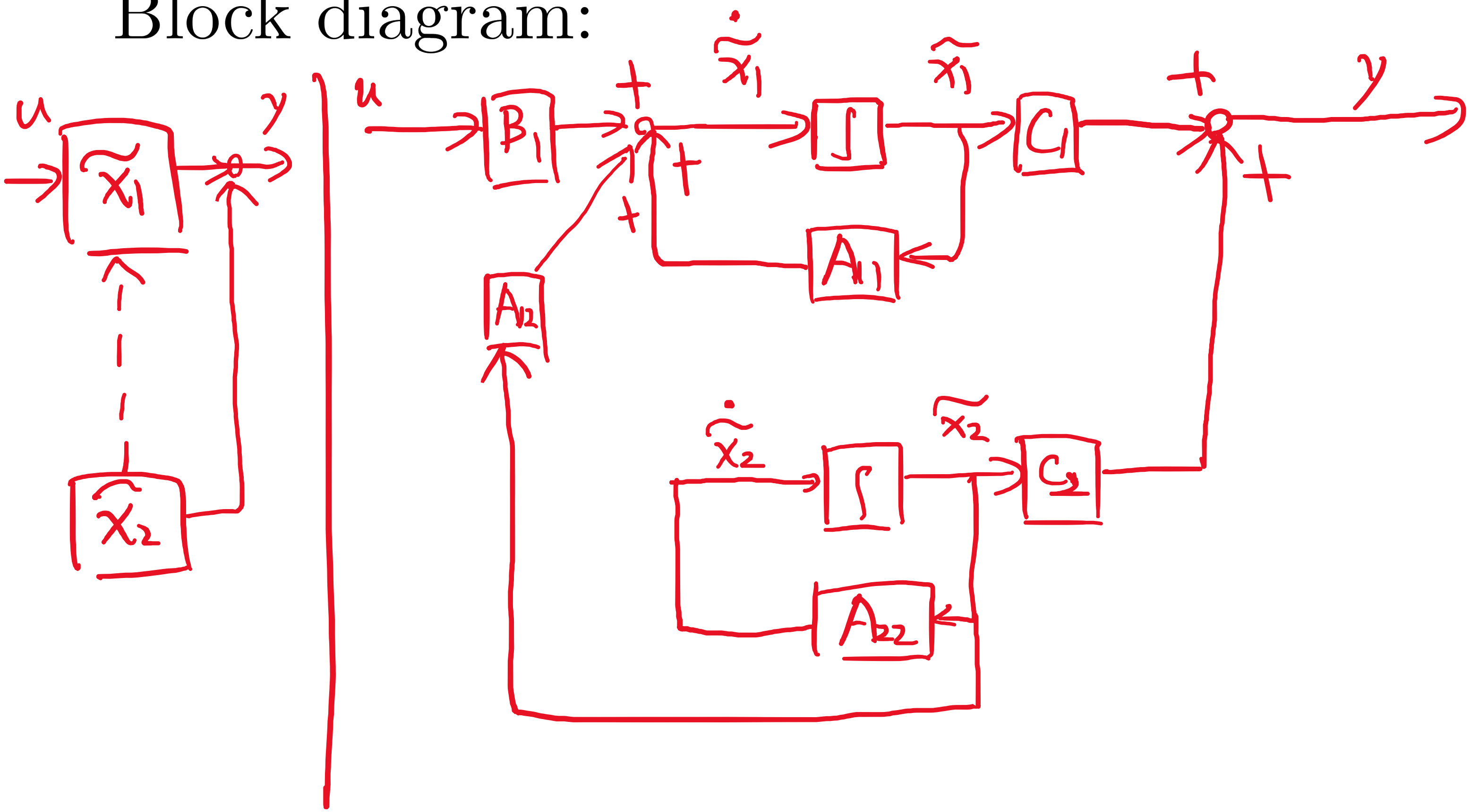
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

(A_{11}, B_1) is controllable

Kalman decomposition

$\dot{x} = Ax + Bu$ can be decomposed into
a controllable part and an uncontrollable part:

Block diagram:



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observable

unobservable

$$y = [C_1 \; 0] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

(C_1, A_{11}) is observable

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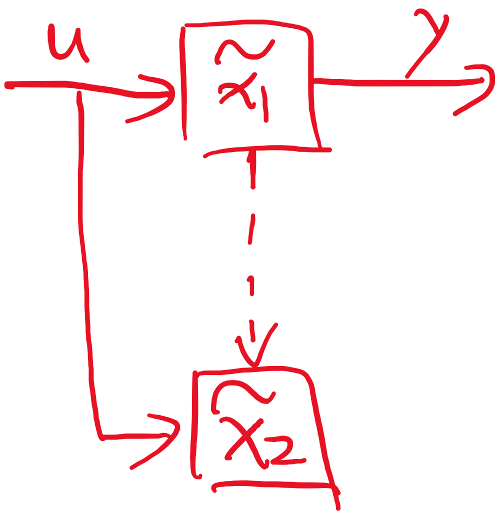

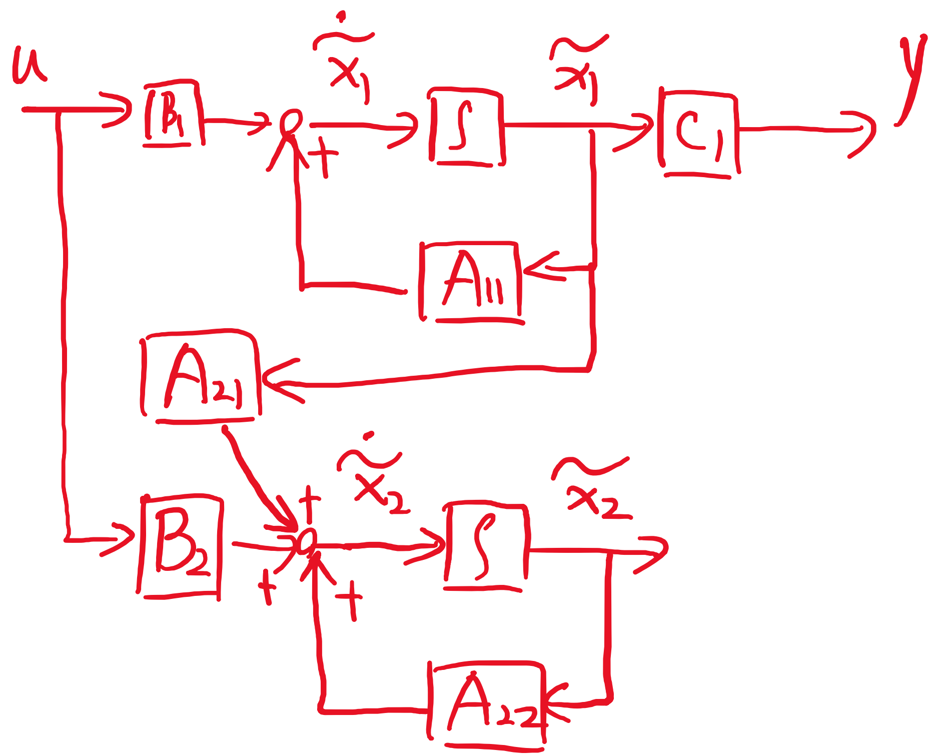

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$\dot{x} = Ax + Bu$, $y = Cx$ can be decomposed into 4 parts:

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There exists an invertible V, with $\tilde{x} := V^{-1}x$
s.t. $\dot{\tilde{x}} = V^{-1}AV\tilde{x} + V^{-1}Bu$, $y = CV\tilde{x}$

$$\begin{array}{l} \text{c + o} \\ \text{c + uo} \\ \text{uc + o} \\ \text{uc + uo} \end{array} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} \boxed{A_{11}} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} \boxed{B_1} \\ B_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \boxed{C_1} & 0 & C_3 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$

Kalman decomposition

$\dot{x} = Ax + Bu$, $y = Cx$ can be decomposed into 4 parts:

1) $\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right)$ is controllable

2) $\left(\begin{bmatrix} C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}\right)$ is observable

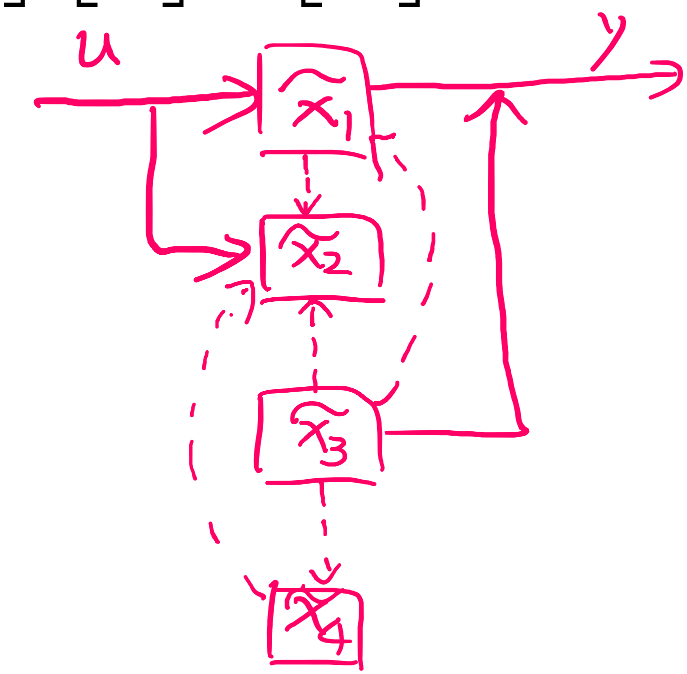
3) (A_{11}, B_1) is controllable, (C_1, A_{11}) is observable

Kalman decomposition

$\dot{x} = Ax + Bu$, $y = Cx$ can be decomposed into 4 parts:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \\ \dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [C_1 \quad 0 \quad C_3 \quad 0] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}$$



$\tilde{x}_1$:

$\tilde{x}_2$:

$\tilde{x}_3$:

$\tilde{x}_4$:

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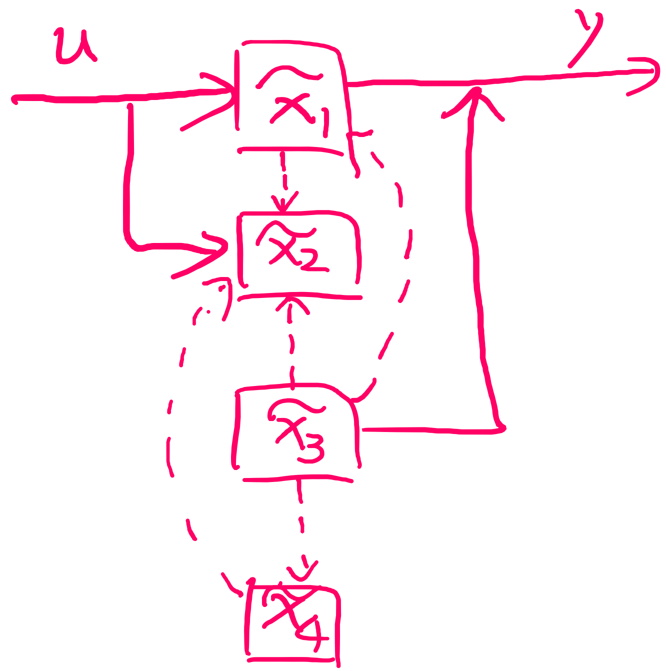

State estimation problem

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
with unknown initial state $x(0)$
Want to estimate $x(t)$, based on $y(0), \dots, y(t)$

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To motivate, consider the simplest state estimator:
$\dot{\hat{x}} = A\hat{x} + Bu$ ($\hat{x}$ is estimate of x)
$\hat{y} = C\hat{x}$ ($\hat{y}$ is estimate of y; y is observed but not used)

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Define the state estimation error $e := \hat{x} - x \rightarrow 0$
Then $\dot{e} := \dot{\hat{x}} - \dot{x} = (A\hat{x} + Bu) - (Ax + Bu)$

$= A(\underline{\hat{x} - x}) + \cancel{(Bu - Bu)}$

$= Ae$