Output Feedback Control

Output feedback control: stabilization

Consider $\dot{x} = Ax + Bu$, $y = Cx$ ($x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$)
and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

Observer-based output feedback control

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable

Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

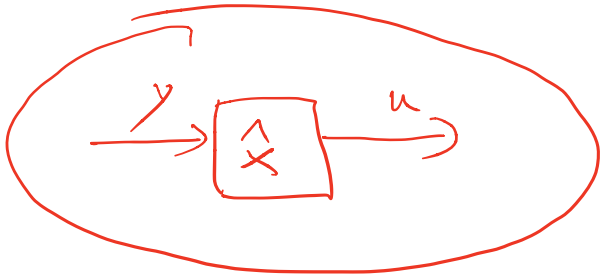
Consider observer-based output feedback control:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$
$$u = F\hat{x}$$

L is s.t. $A + LC$ stable

F is s.t. $A + BF$ stable

$$\begin{cases} \dot{\hat{x}} = (A + BF + LC)\hat{x} - Ly \\ u = F\hat{x} \end{cases}$$



Observer-based output feedback control

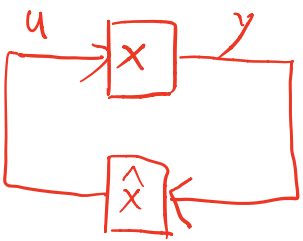
Plant: $\dot{x} = Ax + Bu,\ y = Cx$

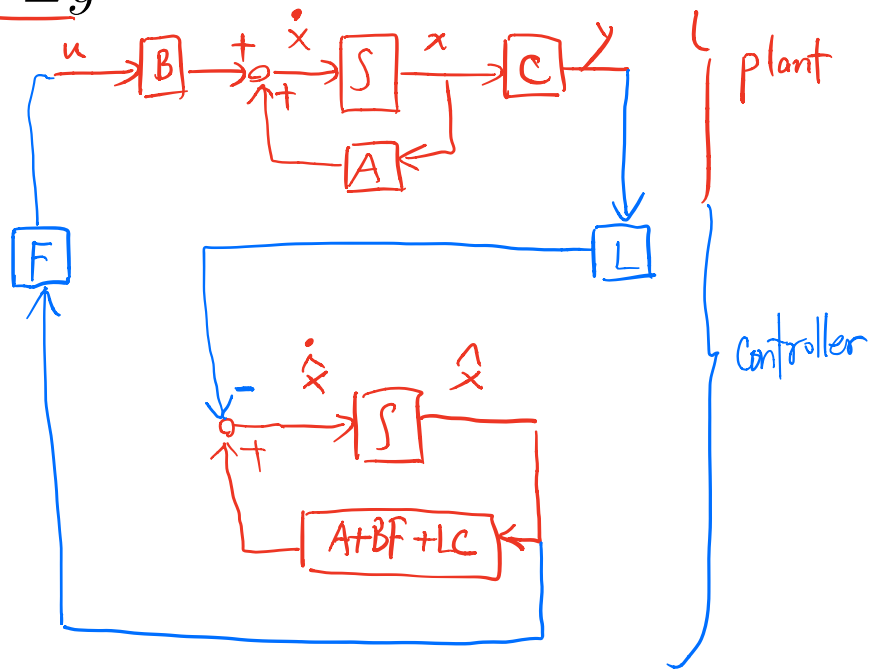
Controller:

$$\dot{\hat{x}} = (A + BF + LC)\hat{x} - Ly$$

$$u = F\hat{x}$$

Block diagram





Observer-based output feedback control

$\boxed{u = F\hat{x}}$

Analysis:

$$\begin{cases} \dot{x} = Ax + Bu = Ax + BF\hat{x} \\ \dot{\hat{x}} = (A + BF + LC)\hat{x} - LCx \end{cases}$$

$\boxed{y = Cx}$

$$\Rightarrow \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix}}_{\in \mathbb{R}^{2n}} = \underbrace{\begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix}}_{\in \mathbb{R}^{2n \times 2n}} \underbrace{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}}_{\in \mathbb{R}^{2n}}$$

"closed-loop system": plant + controller

Is this closed-loop system stable?

$x(t) \to 0$, $\hat{x}(t) \to 0$

Let's analyze the eigenvalues of $\begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix}$

Observer-based output feedback control

Consider a similarity transformation by

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \quad \left(T^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\right)$$

$$TT^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$T^{-1}T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Then $T^{-1}\begin{bmatrix} A & BF \\ -LC & A+BF+LC \end{bmatrix} T =$

$$\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\begin{bmatrix} A & BF \\ -LC & A+BF+LC \end{bmatrix}\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} =$$

$$\begin{bmatrix} A & BF \\ -(A+LC) & A+LC \end{bmatrix}\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} =$$

$$\begin{bmatrix} A+BF & BF \\ 0 & A+LC \end{bmatrix}$$

all eigs. have negative real parts.

Recap

Consider $\dot{x} = Ax + Bu$, $y = Cx$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^p)$
and suppose (A, B) stabilizable, (C, A) detectable
Goal: design u based on y s.t. $x(t) \to 0$ as $t \to \infty$
for all (unknown) initial state $x(0)$

Solution:
Step 1: Design L s.t. $A + LC$ is stable
Step 2: Design F s.t. $A + BF$ is stable
Step 3: Set observer-based output feedback controller:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$
$$u = F\hat{x}$$

Seperation Principle