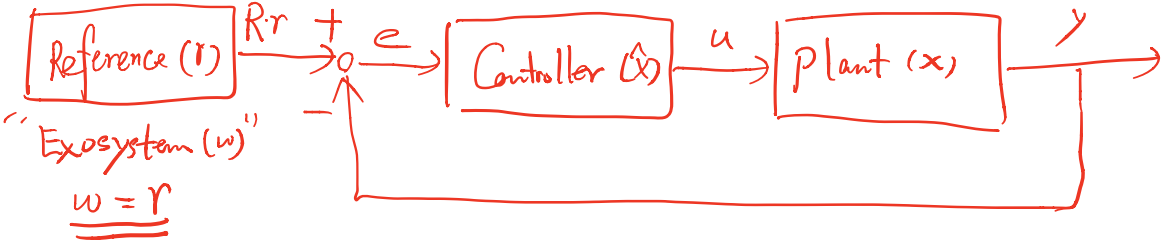
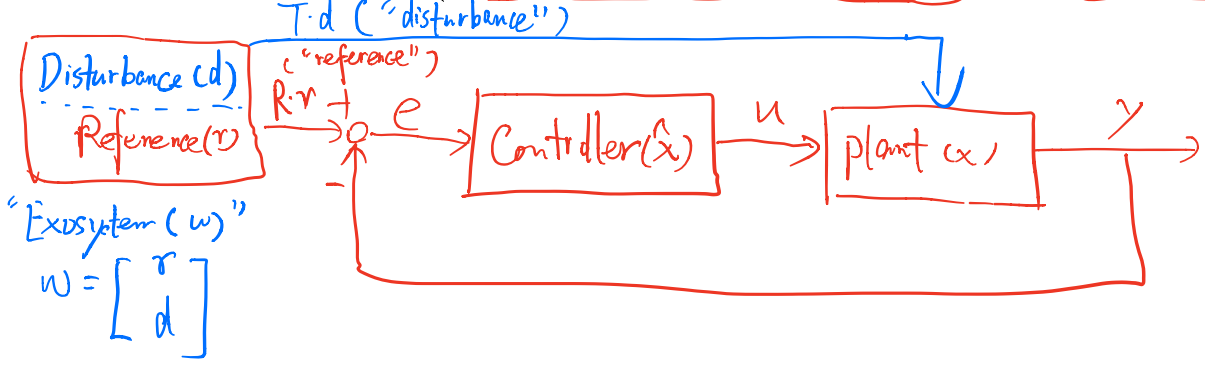
Reference Tracking with Disturbance Rejection: output feedback

So far

Block diagram (reference tracking)



Block diagram (reference tracking + disturbance rejection)



Reference tracking with disturbance rejection

Consider

$$\text{Plant:} \quad \dot{x} = Ax + Bu + Tw$$

$$\text{Exosystem:} \quad \dot{w} = Sw$$

(S has only unstable eigs.)

$$\text{Tracking error} \quad e = Rw - Cx$$

Reference

Disturbance

w : reference + disturbance

Consider "plant+~~reference~~":

exosystem

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$\dot{\tilde{x}}$ $\quad \tilde{A}$ $\quad \tilde{x}$ $\quad \tilde{B}$

$$e = \begin{bmatrix} -C & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

$\tilde{C}$ $\quad \tilde{x}$

Reference tracking with disturbance rejection

Then

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad \tilde{x} = \begin{bmatrix} x \\ w \end{bmatrix}, \; \tilde{A} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} \; \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$e = \tilde{C}\tilde{x}, \qquad \tilde{C} = [-C \; R]$$

Control specifications:

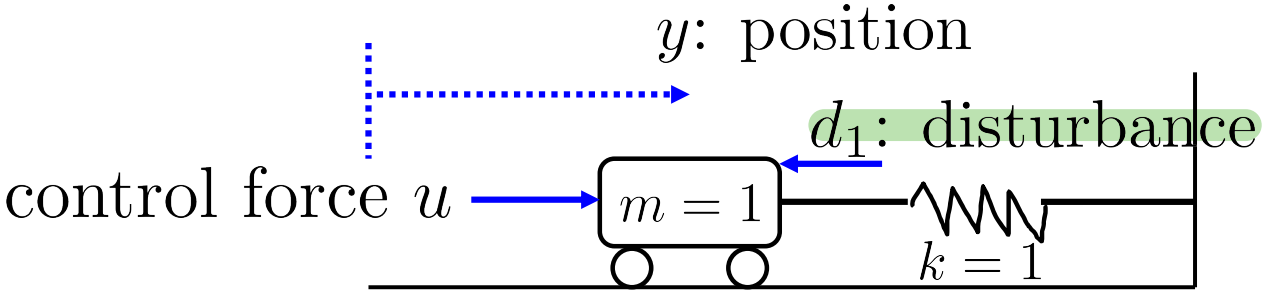
1) If $(\forall t \geq 0) w(t) = 0$, then $(\forall x(0) \in \mathbb{R}^n) \lim_{t\to\infty} x(t) = 0$

"internal stability"

2) $(\forall x(0) \in \mathbb{R}^n, \forall w(0) \in \mathbb{R}^k) \lim_{t\to\infty} e(t) = 0$

"reference tracking + disturbance rejection"

Example: car-spring system



ODE: $m\ddot{y} = u - ky - d_1$

Choose state variables: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Example: car-spring system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d_1, \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Suppose $d_1(t)$ is a sinusoidal disturbance of frequency 10rad/s: $\dot{d} = S_d d$, $d = [d_1 \ d_2]^\top \in \mathbb{R}^2$ $\quad S_d = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix}$

Suppose there is a constant reference signal $r(t) = c$ that we want the output $y(t)$ to track: $\dot{r} = S_r r$, $\quad S_r = 0$

"Exosystem": $w = \begin{bmatrix} r \\ d \end{bmatrix}$

S : $0, 10j, -10j$ (all unstable)

$$\dot{w} = \begin{bmatrix} \dot{r} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} S_r & 0 \\ 0 & S_d \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -100 & 0 \end{array}\right] \begin{bmatrix} r \\ d_1 \\ d_2 \end{bmatrix}$$

Example: car-spring system

Plant:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{T \in \mathbb{R}^{2\times 3}} w = \begin{bmatrix} -\frac{r}{d_1} \\ d_2 \end{bmatrix}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Tracking error:

$$e = -x_1 + r = \underbrace{-\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{R \in \mathbb{R}^{1\times 3}} w = \begin{bmatrix} r \\ d_1 \\ d_2 \end{bmatrix}$$

Example: car-spring system

Putting all together:

$$\dot{x} = Ax + Bu + Tw$$
$$\dot{w} = Sw$$
$$e = Rw - Cx$$

Consider "plant+~~reference~~ exosystem":

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -100 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \cdots \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \\ 0 \end{bmatrix} u$$

$\tilde{A}$ $\tilde{B}$

$$e = \begin{bmatrix} -C & R \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} x \\ w \end{bmatrix}$$

Output-feedback control

Consider $\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$

$$e = [-C \;\; R] \begin{bmatrix} x \\ w \end{bmatrix}$$

Assume only the output e (tracking error) is measurable.

Assume $(\tilde{C}, \tilde{A})$ detectable.

Design an estimator-based output-feedback control:

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e)$$

$$u = F\hat{x}$$

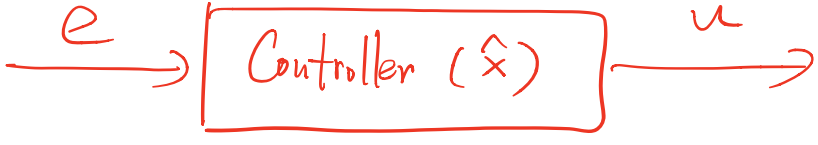
Output-feedback control

Estimator-based output-feedback control:

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{B}u + L(\tilde{C}\hat{x} - e)$$
$$u = F\hat{x}$$

$$\Rightarrow \quad \dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$

Block diagram ("error-feedback control"):



Theorem

Consider $\dot{x} = Ax + Bu + Tw$

$\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

Reference tracking with disturbance rejection problem is solvable by output-feedback control

iff (A, B) stabilizable, $(\tilde{C}, \tilde{A})$ detectable, and $\exists X, U$ s.t.

$$AX - XS + (BU + T) = 0$$
$$-CX + R = 0$$

(regulator equations)

Solution

Consider $\dot{x} = Ax + Bu + Tw$

$\dot{w} = Sw$ (eigenvalues of S are unstable)

$e = Rw - Cx$

When the conditions hold, control design procedure:

1) Compute $F = [F_1 \; F_2]$ as in state-feedback control

separation principle

1.1) Compute F_1 s.t. $A + BF_1$ is stable

1.2) Solve for X, U from $AX - XS + (BU + T) = 0$, $-CX + R = 0$

1.3) Set $F_2 = U - F_1 X$

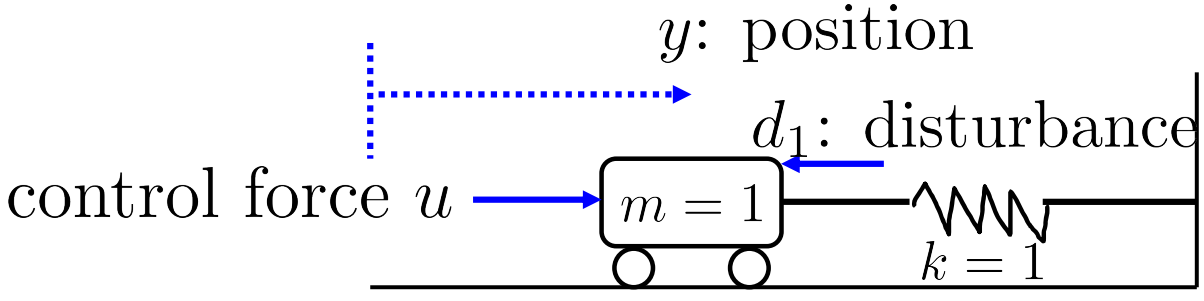
2) Compute L s.t. $\tilde{A} + L\tilde{C}$ is stable

3) Set

$$\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$

$$u = F\hat{x}$$

Example: car-spring system



$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} w$$

(A, B) is stabilizable

1.1) Compute F_1 s.t. eigenvalues of $A + BF_1$ are $-1, -1$

$F_1 = [0 \quad -2]$

Example: car-spring system

1.2) Solve $AX - XS + (BU + T) = 0$

$$-CX + R = 0$$

i.e. Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}_{2\times 3}$ $U = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}_{1\times 3}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} - \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ U = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \end{cases}$$

Example: car-spring system

1.3) Set $F_2 = U - F_1 X = [1 \quad 1 \quad 0] - [0 \quad -2] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= [1 \quad 1 \quad 0] - [0 \quad 0 \quad 0]$

$= [1 \quad 1 \quad 0]$

So $F = [F_1 \; F_2] = \left[\begin{array}{ccc|ccc} 0 & -2 & & 1 & 1 & 0 \end{array}\right]$

Example: car-spring system

$$\tilde{A} = \begin{bmatrix} A & T \\ 0 & S \end{bmatrix} =$$

$$\tilde{C} = [-C \;\; R] =$$

$\operatorname{rank}\begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \tilde{C}\tilde{A}^3 \\ \tilde{C}\tilde{A}^4 \end{bmatrix} = 5$

$(\tilde{C}, \tilde{A})$ observable

2) Compute L s.t. eigenvalues of $\tilde{A} + L\tilde{C}$ are $-1, -1, -1, -1, -1$

$$L = \begin{bmatrix} 5 \\ -91 \\ -0.01 \\ 490 \\ -9005 \end{bmatrix}$$

3) Set

$$\dot{\hat{x}} = (\tilde{A} + \tilde{B}F + L\tilde{C})\hat{x} - Le$$
$$u = F\hat{x}$$