Chapter 8

Optimal Control

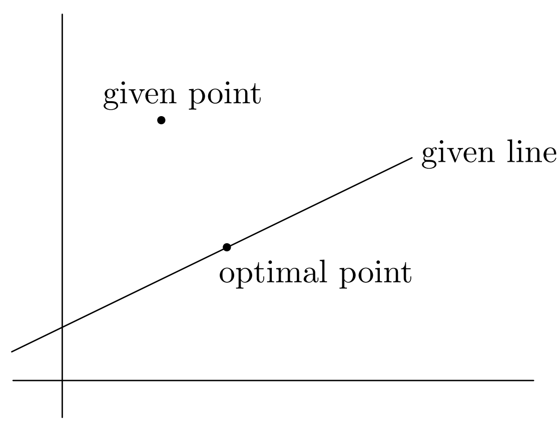
In this chapter we study the simplest optimal control problem, linear quadratic regulation (LQR). There are many approaches to optimal control—dynamic programming, calculus of variations, Pontryagin's maximum principle, and others. But none is entirely just right for the LQR problem. At some point you have to wave your hands in the derivation of the solution. After that, there are rigorous proofs that the controller you derived by waving your hands really is optimal.

We're going to adopt the method of Lagrange multipliers for the hand-waving part. This is quite interesting and useful in its own right.

8.1 Minimizing Quadratic Functions with Equality Constraints

The optimal control problem that we'll solve involves minimizing a quadratic function with an equality constraint. Let's begin with a very simple such example:

Example In the plane, find the point on a given line that is closest to a given point:



Obviously, you can get the closest point by drawing the perpendicular from the given point to the given line.

Before we solve this problem, let's clarify some notation. The norm of $x = (x_1, x_2)$ is

$$\|x\| = \left(x_1^2 + x_2^2\right)^{1/2},$$

and this can also be written $\|x\| = (x^T x)^{1/2}$, that is,

$$x^T x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2.$$

To develop a solution method, suppose the given point is $v = (1, 2)$ and the equation of the given line is

$$x_2 = 0.5x_1 + 0.2.$$

Let $x = (x_1, x_2)$ be the point being sought. Define

$$c^T = \begin{bmatrix} -0.5 & 1 \end{bmatrix}, \quad b = 0.2.$$

Then x is on the line iff $c^T x = b$. Also, the distance from v to x is $\|v - x\|$. Note that $\|v - x\|$ is minimum iff $\|v - x\|^2$ is minimum. Thus we have arrived at the following equivalent problem: minimize the quadratic function $\|v - x\|^2$ of x subject to the equality constraint $c^T x = b$. Notice that

$$\|v - x\|^2 = (v - x)^T (v - x) = v^T v - v^T x - x^T v + x^T x.$$

The right-hand side is a quadratic function of x. Since $x^T v = v^T x$ (dot product of real vectors is symmetric), we have

$$\|v - x\|^2 = v^T v - 2v^T x + x^T x.$$

So we've reduced the problem to

$$\min_{x,\ c^T x = b} v^T v - 2v^T x + x^T x.$$

We'll return to this after we review some calculus.

Aside: This specific problem is easy to solve this way: Substitute the constraint $x_2 = 0.5x_1 + 0.2$ into

$$(1 - x_1)^2 + (2 - x_2)^2,$$

to get a function $f(x_1)$. Set the derivative of f to zero, solve for x_1, then get x_2. The answers are $x_1 = 1.52$, $x_2 = 0.96$. □

Jacobians

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function. Thus, in the expression $f(x)$, x is a vector with n components and f is a vector with m components. So we can write

$$x = (x_1, \ldots, x_n), \quad f = (f_1, \ldots, f_m).$$

The Jacobian of f, denoted $\dfrac{\partial f}{\partial x}$, is the $m \times n$ matrix whose ij^{th} element is $\partial f_i / \partial x_j$. If $m = 1$, then $\dfrac{\partial f}{\partial x}$ is a row vector, the gradient of f, usually written ∇f.

Another way to think of the Jacobian is via the directional derivative. The derivative of f at the point x in the direction of the vector h is defined to be

$$\frac{d}{d\varepsilon} f(x+\varepsilon h)\bigg|_{\varepsilon=0}.$$

This turns out to be a linear function of the vector h, and it must therefore equal Mh for some matrix M. In fact, M equals the Jacobian of f at x.

Example

$$m=1,\ n=2,\ f(x)=c_1x_1+c_2x_2,\quad \frac{\partial f}{\partial x}=\left[\begin{array}{cc} c_1 & c_2 \end{array}\right]$$

More generally, if $f(x)=c^Tx$, then $\dfrac{\partial f}{\partial x}=c^T$. This can be derived like this:

$$\begin{aligned} f(x+\varepsilon h) &= c^T(x+\varepsilon h)\\ &= c^Tx+\varepsilon c^Th \end{aligned}$$

$$\frac{d}{d\varepsilon} f(x+\varepsilon h)=c^Th$$

$$\frac{d}{d\varepsilon} f(x+\varepsilon h)\bigg|_{\varepsilon=0}=c^Th$$

$$\frac{\partial f}{\partial x}=c^T.$$

□

Example If

$$f(x)=\|x\|^2=x_1^2+\cdots+x_n^2,$$

then $\dfrac{\partial f}{\partial x}=2x^T$. More generally, consider $f(x)=x^TQx$, where Q is a symmetric matrix. You can derive that $\dfrac{\partial f}{\partial x}=2x^TQ$. □

Example If $f(x)=x(\|x\|^2-1)$, $f:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, then

$$\begin{aligned} f(x+\varepsilon h) &= (x+\varepsilon h)(\|x+\varepsilon h\|^2-1)\\ &= (x+\varepsilon h)(\|x\|^2+2\varepsilon x^Th+\varepsilon^2\|h\|^2-1)\\ &= x\|x\|^2+\varepsilon\|x\|^2h+2\varepsilon xx^Th-\varepsilon h+\text{HOT}. \end{aligned}$$

Thus

$$\frac{\partial f}{\partial x}=(\|x\|^2-1)I+2xx^T.$$

□

Lagrange Multipliers

Now we return to the first example in this section. It had the form

$$\min_{x,\ c^T x = b} f(x), \quad f(x) = v^T v - 2v^T x + x^T x, \quad c^T = \begin{bmatrix} -0.5 & 1 \end{bmatrix}, \quad b = 0.2$$

We are going to use the method of **Lagrange multipliers**. The idea is to absorb the constraint $c^T x = b$, or equivalently $c^T x - b = 0$, into the function being minimized, leaving an unconstrained problem. Define the **Lagrangian**

$$L(x, \lambda) = f(x) + \lambda(c^T x - b).$$

Here λ is an unknown that multiplies the constraint equation. It turns out a necessary condition for optimality of x is that L should be stationary with respect to both x and λ, that is,

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0.$$

These two equations give

$$\frac{\partial f}{\partial x} + \lambda c^T = 0, \quad c^T x - b = 0,$$

or, using the form of f,

$$-2v^T + 2x^T + \lambda c^T = 0, \quad c^T x - b = 0.$$

Finally, taking transpose and rearranging, we have

$$2x + \lambda c = 2v, \quad c^T x = b.$$

These can be assembled into one equation:

$$\begin{bmatrix} 2I & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 2v \\ b \end{bmatrix}.$$

Let's put in our values for v, c, b:

$$\begin{bmatrix} 2 & 0 & -0.5 \\ 0 & 2 & 1 \\ -0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0.2 \end{bmatrix}.$$

This has a unique solution because the matrix is invertible:

$$x = (1.52, 0.96), \quad \lambda = 2.08.$$

The x is the optimal x, the closest point, and the λ can be discarded—it was introduced only to solve the problem. □

Let's look at a somewhat more general problem by the Lagrange multiplier method.

Example We'll solve the problem

$$\text{minimize}_x \|c - Ax\|$$

subject to the constraint $Bx = d$. Here x, c, d are vectors and A, B matrices. Assume A has full column rank and B has full row rank.

Define

$$\begin{aligned} J(x) = \|c - Ax\|^2 &= (c - Ax)^T(c - Ax) \\ &= c^Tc - c^TAx - x^TA^Tc + x^TA^TAx \\ &= c^Tc - 2c^TAx + x^TA^TAx \end{aligned}$$

and

$$L(x, \lambda) = J(x) + \lambda^T(Bx - d).$$

Here the Lagrange multiplier has to be a vector. Differentiating with respect to x then λ, we get

$$-2c^TA + 2x^TA^TA + \lambda^TB = 0, \quad Bx - d = 0.$$

Transposing the first gives

$$-2A^Tc + 2A^TAx + B^T\lambda = 0, \quad Bx - d = 0.$$

Collect as one equation:

$$\begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^Tc \\ b \end{bmatrix}.$$

If it can be proved that the matrix on the left is invertible, then the optimal x is

$$x = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2A^Tc \\ b \end{bmatrix}.$$

So let's see that the matrix

$$\begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix}$$

is invertible. It suffices to prove that the only solution to the homogeneous equation

$$\begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0$$

is the trivial solution. So start with

$$\begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0.$$

Thus

$$2A^TAx + B^T\lambda = 0, \quad Bx = 0.$$

Since A has full column rank, the matrix A^TA is positive definite, hence invertible. Thus

$$x + (2A^TA)^{-1}B^T\lambda = 0, \quad Bx = 0.$$

Multiply the first equation by B and use the second:

$$B(2A^TA)^{-1}B^T\lambda = 0.$$

Pre-multiply by λ^T:

$$\lambda^TB(2A^TA)^{-1}B^T\lambda = 0.$$

Since $(2A^TA)^{-1}$ is positive definite, it follows that $B^T\lambda = 0$. Then, since B^T has full column rank, $\lambda = 0$. Finally, from the equation

$$x + (2A^TA)^{-1}B^T\lambda = 0,$$

we get that $x = 0$. Thus $x = 0, \lambda = 0$ is the only solution of

$$\begin{bmatrix} 2A^TA & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0.$$

□

Why the Lagrange multiplier method works

Consider the problem of minimizing a function $f(x)$ subject to an equality constraint $g(x) = 0$.

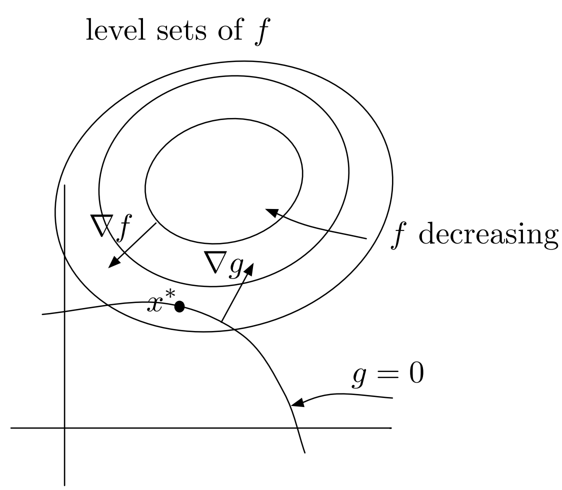
To be able to draw pictures, let's suppose

$$f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

The set of all x satisfying the constraint $g(x) = 0$ typically is a curve. For a given constant c, the set of all x satisfying $f(x) = c$ is called a level set of f. Now assume x^* is a locally optimal point for the problem $\min_{g(x)=0} f(x)$. That is, if x is nearby x^* and $g(x) = 0$, then $f(x) > f(x^*)$.

Claim The gradients $\nabla f(x^*)$, $\nabla g(x^*)$ are collinear.

Proof The picture near x^* looks like this:



From this, the claim is clear. □

Thus there is a scalar λ^* such that $\nabla f(x^*) + \lambda^* \nabla g(x^*) = 0$. This implies the gradient of the function

$$f(x) + \lambda^* g(x)$$

equals zero at x^*. Finally, this implies the Lagrangian

$$L(x, \lambda) = f(x) + \lambda g(x)$$

satisfies

$$\frac{\partial L}{\partial x}(x^*, \lambda^*) = 0, \quad \frac{\partial L}{\partial \lambda}(x^*, \lambda^*) = 0.$$

In conclusion, a necessary condition for a point x^* to be a local optimum for the problem $\min_{g(x)=0} f(x)$ is that there exist a point λ^* such that the derivative of the Lagrangian $L(x, \lambda)$ equals zero at x^*, λ^*.

8.2 The LQR Problem and Solution

As with several other control problems (e.g., controllability), the LQR problem is posed initially as a rather idealized. formal problem, and then used in applications in a different way (controllability is posed as reachability and then used as a condition for arbitrary pole assignment). So be prepared for a formal problem statement.

The **Linear Quadratic Regulator** (LQR) **Problem** can be stated as follows. We consider the usual state-space model:

$$\begin{aligned} x(0) &= x_0 \\ \dot{x} &= Ax + Bu. \end{aligned}$$

The initial state, x_0, is fixed and the desired state is zero. More specifically, the goal is to have $x(t) \longrightarrow 0$ optimally, in some sense. So at time t, $\|x(t)\|$ is a measure of the error—the distance from $x(t)$ to the origin. The integral-squared error is

$$\int_0^\infty \|x(t)\|^2 dt.$$

Note that

$$\|x(t)\|^2 = x(t)^T x(t).$$

A somewhat more general measure of error is $x(t)^T Q x(t)$, where Q is symmetric and positive semidefinite. E.g.,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Aside A symmetric matrix Q is **positive semidefinite** if

$$x^T Q x \geq 0, \quad \forall x.$$

It is a fact that the eigenvalues of a symmetric matrix are all real, and a symmetric matrix is positive semidefinite iff all its eigenvalues are ≥ 0. E.g.,

$$Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is positive semidefinite. Eigs: 0, 1.8769, 10.1231. A symmetric matrix Q is **positive definite** if

$$x^T Q x > 0, \quad \forall x \neq 0.$$

It is a fact that a symmetric matrix is positive definite iff all its eigenvalues are > 0. E.g.,

$$Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 10 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

is positive definite. Eigs: 1.8695, 3.8505, 10.2800.

In this way we're led to the problem: Given A, B, Q, x_0 and

$$\begin{aligned} x(0) &= x_0 \\ \dot{x} &= Ax + Bu, \end{aligned}$$

find u to minimize

$$\int_0^\infty x(t)^T Q x(t) dt.$$

The trouble with this problem formulation is that u will want to be unbounded. So a better cost to minimize is

$$J = \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t) \right] dt,$$

where R is symmetric, positive definite; e.g., $R = I$. As we will see, it turns out under some mild technical assumptions that the sequence that minimizes J has the form $u(t) = Fx(t)$, that is, the optimal control law is state feedback. Furthermore, $A + BF$ is stable. So the solution of the LQR problem provides an alternative way to stabilize an unstable plant; in fact, this way is more sound numerically than eigenvalue assignment.

The matrix F is uniquely determined by the data (A, B, Q, R). The MATLAB command is

$$F = -\mathrm{lqr}(A, B, Q, R)$$

Typically, Q and R are used as design parameters: One proceeds as follows:

1. Choose any Q, R.

2. Compute F by solving the LQR problem.

3. Simulate the controlled system.

4. To improve the response, modify Q, R and return to step 2.

It is interesting to note that the cost function J can be reformulated in terms of an artificial output. We have

$$x^T Q x + u^T R u = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.$$

Being positive semidefinite, Q and R have positive semidefinite square roots. Defining

$$C = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix},$$

we get

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}.$$

The further definition

$$y = Cx + Du,$$

gives

$$\begin{aligned} x^T Q x + u^T R u &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &= y^T y \end{aligned}$$

and therefore

$$J = \int_0^\infty y(t)^T y(t) dt = \int_0^\infty \|y(t)\|^2 dt.$$

To see where we're going, we'll now give the solution of the LQR problem. The assumptions are these:

1. Q is positive semidefinite and R is positive definite.

2. (A, B) is stabilizable.

3. (Q, A) is detectable.

Then the optimal control law is $u = Fx$, where F is defined as follows:

1. Form the matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}.$$

   This is called the Hamiltonian matrix.

2. It turns out that H has no eigenvalues on the imaginary axis, n eigenvalues in $\Re\ s < 0$, and n in $\Re\ s > 0$. Thus its Jordan form, H_{JF}, must, after possible re-ordering of blocks, have the form

$$H_{JF} = \begin{bmatrix} H^- & 0 \\ 0 & H^+ \end{bmatrix},$$

   where H^- has all eigenvalues in $\Re\ s < 0$ and H^+ has all eigenvalues in $\Re\ s > 0$. That is, there exists an invertible $2n \times 2n$ matrix V satisfying

$$V^{-1}HV = \begin{bmatrix} H^- & 0 \\ 0 & H^+ \end{bmatrix}.$$

   Partition V as

$$V = \begin{bmatrix} V^- & V^+ \end{bmatrix},$$

   each part being $2n \times n$.

3. Now partition V^- as

$$V^- = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

   where $X_1, X_2 \in \mathbb{R}^{n \times n}$. It turns out that X_1 is nonsingular. Define $X := X_2 X_1^{-1}$.

4. Then $F = -B^T R^{-1} X$.

Example A very simple example is

$$A = B = Q = R = 1.$$

Then

$$H = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The MATLAB command

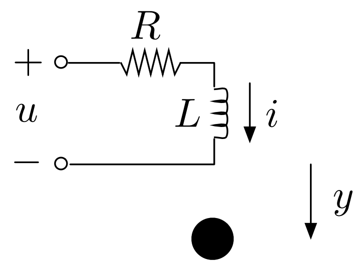
$$[V, H_{JF}] = \mathrm{eig}(H)$$

yields

$$V = \begin{bmatrix} -0.3827 & -0.9239 \\ -0.9239 & 0.3827 \end{bmatrix}, \quad H_{JF} = \begin{bmatrix} -1.414 & 0 \\ 0 & 1.414 \end{bmatrix}.$$

Thus $X_1 = -0.3827, X_2 = -0.9239, X = 2.414, F = -2.414$. □

Example Maglev. Let's consider again the magnetic levitation system. The schematic diagram and equations are as follows:



We took the numerical values $M = 0.1$ Kg, $R = 15$ ohms, $L = 0.5$ H, $K = 0.0001$ Nm^2/A^2, $g = 9.8$ m/s^2. The problem is to design a controller that will stabilize the ball at $y = 0.01$ m.

Define state variables $x = (x_1, x_2, x_3) = (i, y, \dot{y})$. Then the linearized model is

$$\dot{\delta x} = A\delta x + B\delta u, \quad \delta y = C\delta x,$$

where δ denotes displacement away from equilibrium, and

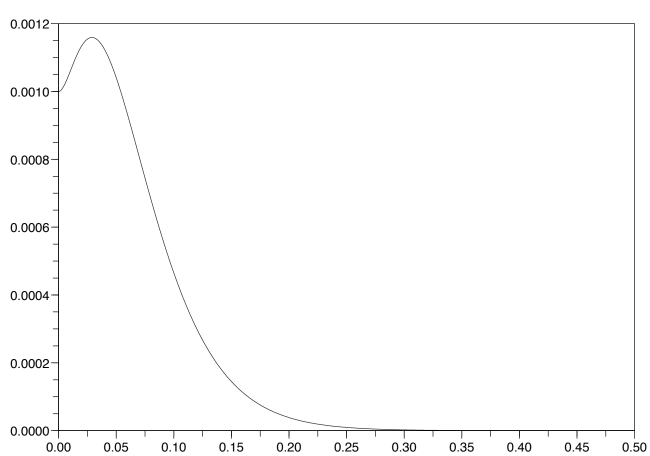
$$A = \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -19.8 & 1940 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

As a base, since we're trying to control y, start with

$$Q = \mathrm{diag}(0, 1, 0), \quad R = I.$$

Here R is 1×1. The next graph shows a plot of $\delta y(t)$ versus t, where

$$\dot{\delta x} = (A + BF)\delta x, \quad \delta y = C\delta x, \quad \delta x(0) = (0, 0.001, 0).$$



The feedback gains are

$$F = \left[\begin{array}{ccc} -44.3 & 7352 & 166.1 \end{array} \right].$$

If we want to alter the response, we play with Q (no point in playing with R, because it's a scalar). Obviously we have to do a full nonlinear simulation of all signals to see what's feasible. □

8.3 Hand Waving

The solution involves the Hamiltonian matrix

$$H = \left[\begin{array}{cc} A & -BR^{-1}B^T \\ -Q & -A^T \end{array} \right].$$

Where on earth does this come from? This section gives some motivation for how it arises.

It's convenient to scale J by $1/2$:

$$J = \frac{1}{2}\int_0^\infty [x(t)^T Qx(t) + u(t)^T Ru(t)]dt.$$

Write the plant model in the form

$$-\dot{x} + Ax + Bu = 0.$$

We intend to think of the problem as minimizing J subject to this equality constraint. The method of Lagrange multipliers is perfectly suited for such a problem. So we define a new function, denoted L for Lagrangian:

$$L = \int_0^\infty \left\{ \frac{1}{2}x(t)^T Qx(t) + \frac{1}{2}u(t)^T Ru(t) + \lambda(t)^T[-\dot{x}(t) + Ax(t) + Bu(t)] \right\} dt.$$

That is, we have added the equality constraint into the cost function; the equality constraint has a multiplier λ, which enters as a dot product. We regard L as a function of x, u, λ.

We get necessary conditions for optimality by differentiating the integrand with respect to x, u, λ. First, λ:

$$-\dot{x} + Ax + Bu = 0.$$

Next, u:

$$u^T R + \lambda^T B = 0.$$

Solving this equation we get $u(t) = -R^{-1}B^T\lambda(t)$. Substituting this into the first equation gives

$$\dot{x} = Ax - BR^{-1}B^T\lambda.$$

Finally, to differentiate L with respect to x, we get rid of $\dot{x}$ by integrating by parts, ignoring the term $\lambda(\infty)x(\infty) - \lambda(0)x(0)$:

$$L = \int_0^\infty \left\{ \frac{1}{2}x(t)^T Qx(t) + \frac{1}{2}u(t)'Ru(t) + \dot{\lambda}(t)^T x(t) + \lambda(t)^T[Ax(t) + Bu(t)] \right\} dt.$$

Then

$$x^TQ + \dot{\lambda}^T + \lambda^T A = 0,$$

i.e.,

$$\dot{\lambda} = -Qx - A^T\lambda.$$

Combining the $\dot{x}$ and $\dot{\lambda}$ equations, we get

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}. \tag{8.1}$$

And, voilà, there's H!

To recap, the matrix H arises in a Lagrangian formulation of the optimal control problem. We argued that if u is an optimal control, then $u = -R^{-1}B^T\lambda$, where x and λ satisfy (8.1).

Let us return to the definition of J:

$$J = \frac{1}{2}\int_0^\infty \left[x(t)^TQx(t) + u(t)^TRu(t)\right] dt.$$

Since R is positive definite, for J to be finite, it seems reasonable that u should satisfy $u(t) \to 0$ as $t \to \infty$. From

$$u = -R^{-1}B^T\lambda',$$

a sufficient condition for this is that $\lambda(t) \to 0$. We shall also impose that $x(t) \to 0$ as $t \to \infty$.

Recap We've argued that if $u(t)$ is an optimal control signal for the LQR problem, then it has the form $u = -R^{-1}B^T\lambda$ where λ is a companion signal (a Lagrange multiplier) to x that together are stable solutions of the equation (8.1). The Lagrange multiplier is also known as the **co-state**. So the optimal state, co-state, and control are defined by the conditions

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$u = -R^{-1}B^T\lambda$$

$$x(t), \lambda(t) \longrightarrow 0.$$

Define $w = (x, \lambda)$ so that

$$\dot{w} = Hw.$$

How can we characterize the stable solutions? Answer: The solution $w(t)$ converges to 0 iff $w(0)$ is in the stable eigenspace of H, namely, Im V^-. But

$$\text{Im } V^- = \text{Im} \begin{bmatrix} I \\ X \end{bmatrix} X_1 = \text{Im} \begin{bmatrix} I \\ X \end{bmatrix}.$$

Thus

$$w(0) \in \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}.$$

Since eigenspaces are invariant subspaces

$$w(t) \in \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix}, \quad \forall t.$$

Thus

$$\begin{bmatrix} -X & I \end{bmatrix} w(t) = 0$$

and hence $\lambda(t) = Xx(t)$. Finally

$$u = -R^{-1}B^T\lambda = u = -R^{-1}B^T Xx,$$

so $F = -R^{-1}B^T X$.

8.4 Sketch of Proof that F is Optimal

Now that we've arrived at a formula for a feedback matrix F, we turn to the proof that $u = Fx$ actually is the optimal control. We're going to skip the proofs that $\sigma(H)$ is symmetric about the imaginary axis, that it has no imaginary eigenvalues, that X_1 is invertible, and that X is positive semidefinite.

We have

$$V^{-1}HV = \begin{bmatrix} H^- & 0 \\ 0 & H^+ \end{bmatrix},$$

and so

$$HV^- = V^- H^-.$$

Thus

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} H^-$$

and hence

$$H \begin{bmatrix} I \\ X \end{bmatrix} X_1 = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H^-$$

so finally

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} X_1 H^- X_1^{-1}. \tag{8.2}$$

Lemma 8.4.1 *X satisfies the algebraic Riccati equation*

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

and $A - BR^{-1}B^T X$ is stable.

Proof Start with (8.2). Pre-multiply by $\begin{bmatrix} X & -I \end{bmatrix}$:

$$\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = 0.$$

This is precisely the Riccati equation. Pre-multiply (8.2) by $\begin{bmatrix} I & 0 \end{bmatrix}$ to get

$$A - BR^{-1}B^T X = X_1 H^- X_1^{-1}.$$

Thus $A - BR^{-1}B^T X$ is stable because H^- is. $\square$

Theorem 8.4.1 *The control signal that minimizes J is $u = Fx$ and it is the unique optimal control.*

Proof The proof is a trick using the completion of a square. Let u be an arbitrary control input for which J is finite. We shall differentiate the quadratic form $x(t)'Xx(t)$ along the solution of the plant equation. To simplify notation, we suppress dependence on t. We have

$$\begin{aligned}
\frac{d}{dt}(x^T Xx) &= \dot{x}^T Xx + x^T X\dot{x} \\
&= (Ax + Bu)^T Xx + x^T X(Ax + Bu) \\
&= x^T (A^T X + XA)x + 2u^T B^T Xx \\
&= x^T (XBR^{-1}B^T X - Q)x + 2u^T B^T Xx \text{ from the Riccati equation} \\
&= -x^T Qx + x^T XBR^{-1}B^T Xx + 2u^T B^T Xx \\
&= -x^T Qx + x^T XBR^{-1}B^T Xx + 2u^T B^T Xx + (u^T Ru - u^T Ru) \\
&\quad \text{—this was the completion of squares trick} \\
&= -x^T Qx - u^T Ru + \|R^{-1/2}B^T Xx + R^{1/2}u\|^2.
\end{aligned}$$

Rearranging terms we have

$$x^T Qx + u^T Ru = -\frac{d}{dt}(x^T Xx) + \|R^{-1/2}B^T Xx + R^{1/2}u\|^2.$$

Now integrate from $t = 0$ to $t = \infty$ and use the fact (not proved) that for J to be finite, $x(t)$ must go to zero:

$$J = x_0^T Xx_0 + \int_0^\infty \|R^{-1/2}B^T Xx + R^{1/2}u\|^2 dt.$$

Thus J is minimum iff $R^{-1/2}B^T Xx + R^{1/2}u \equiv 0$, i.e., $u = Fx$. $\square$

The LQR solution provides a very convenient way to stabilize an LTI plant. Given A, B, select Q, R with $Q \geq 0$, (Q, A) detectable, and $R > 0$. Then the optimal F stabilizes $A + BF$. This is the preferred method over pole assignment.

8.5 Problems

1. **Dynamic programming** (DP) is a clever solution to certain types of optimization problems. As an example, let $\{x_1, \ldots, x_n\}$ be a finite sequence of real numbers and consider the problem $\min_i x_i$ of finding the minimum. The DP method of solving involves defining the **value function**

$$V(i) = \min \{x_i, \ldots, x_n\},$$

that is, $V(i)$ is the minimum "cost-to-go" starting at x_i. The value $V(1)$ is what we seek. Of course, $V(n) = x_n$. Suppose we know $V(i)$ for some i, $1 < i < n$. Then

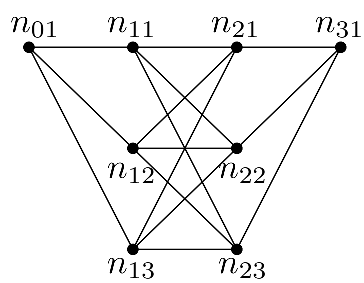
$$\begin{aligned} V(i-1) &= \min \{x_{i-1}, \ldots, x_n\} \\ &= \min \{x_{i-1}, V(i)\}. \end{aligned}$$

Thus the DP algorithm is

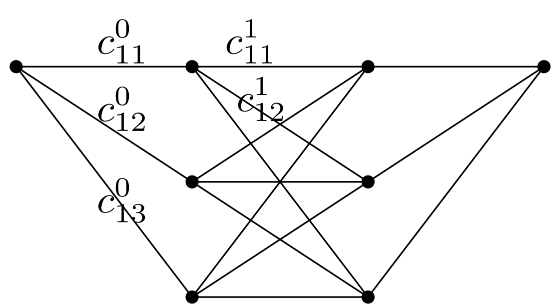
$V(n) = x_n$
for $i = n-1, n-2, \ldots, 1$: $V(i-1) = \min \{x_{i-1}, V(i)\}$

Thus the minimization problem is reduced to a recursion of small minimization problems over just pairs of numbers. Write a MATLAB script for this problem and try an example.

2. Another application of DP is to find a minimum-cost path through a graph. Consider this graph:



The nodes are labeled n_{ij}, where i is interpreted as the stage and j as the node number at that stage. Thus there's one node at stage 0, three nodes at stage 1, etc. One wants to travel from the start node n_{01} to the end node n_{31} with minimum cost. Each link has a cost, labeled like this (not all are shown):

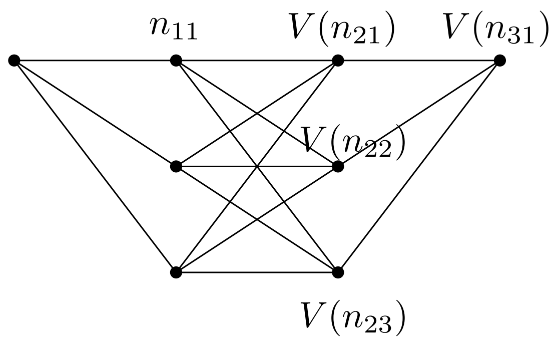


Thus, c_{ij}^k is the cost from node i at stage k to node j at stage $k+1$. The cost of a path is defined to be the sum of the costs of the links.

We define the value function, a real-valued function of the nodes, as follows: $V(n_{ij})$ is the minimum cost to go from node n_{ij} to the end node. The value function at stage 3 is obviously 0. Thus $V(n_{31}) = 0$. The value function at stage 2 is obviously just the cost of the last link:

$$V(n_{21}) = c_{11}^2, \quad V(n_{22}) = c_{21}^2, \quad V(n_{31}) = c_{31}^2.$$

We label these at the nodes:



Now to the value function at stage 1. We will invoke the so-called **principle of optimality**: Consider an optimal path from n_{01} to n_{31}; if this path goes through node n_{1j} at stage 1, then the subpath from node n_{1j} to n_{31} is optimal too. That is, for every optimal path, the cost-to-go is minimum at each point along the path. Note that we're not saying the initial subpath is optimal, but rather the cost-to-go is. Thus at node n_{11}, since there are just three links out, we have

$$V(n_{11}) = \min\ \{c_{11}^1 + V(n_{21}), c_{12}^1 + V(n_{22}), c_{13}^1 + V(n_{23})\}.$$

After the other values are computed at stage 1, one computes $V(n_{01})$, which equals the minimum cost path from start to end. After the value function is computed at every node, it's easy to find optimal paths by moving left to right. Try an example.

The end