Chapter 7

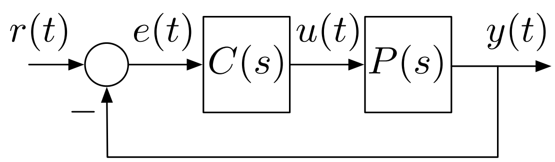
Tracking and Regulation

Now we treat the problem of tracking a reference and/or rejecting a disturbance. The reference and disturbance signals are assumed to satisfy known differential equations, and only asymptotic tracking or disturbance rejection is required—transient response is not an explicit specification.

7.1 Review of Tracking Steps

Besides feedback stability, another common control requirement is to be able to track a constant reference signal. A familiar example is cruise control. We set the desired car speed and then require the car to maintain that constant speed.

We continue with the block diagram



Suppose $r(t)$ is an arbitrary constant value and we require $y(t)$ to converge to that constant value. By linearity, it suffices to meet this requirement for just $r(t) = 1$. So the problem is: Given $r(t) = 1$, design $C(s)$ to achieve feedback stability and $\lim_{t\to\infty} y(t) = 1$, or equivalently, $\lim_{t\to\infty} e(t) = 0$.

Let's address this problem using transfer function methods. The Laplace transform of $r(t) = 1$ is $R(s) = 1/s$. Let $G(s)$ denote the closed-loop transfer function from r to the tracking error e. Then

$$E(s) = G(s)\frac{1}{s}.$$

By the final-value theorem, $\lim_{t\to\infty} e(t) = 0$ iff all the poles of $G(s)$ lie in $\Re\, s < 0$ and $G(0) = 0$. Having all the poles of $G(s)$ lie in $\Re\, s < 0$ will follow from the requirement of feedback stability. Since

$$G = \frac{1}{1+PC},$$

the condition $G(0) = 0$ is equivalent to the condition that $P(s)C(s)$ has a pole at $s = 0$. If P has a pole at $s = 0$, then C merely has to stabilize the feedback loop. If P does not have a pole at $s = 0$, then C must have one. If, furthermore, P has a zero at $s = 0$, then the problem isn't solvable; remember: there can be no unstable pole-zero cancellation. Here's an example where the problem is not solvable: a cart/pendulm where y is the angle of the pendulum. Obviously the angle cannot be maintained at a nonzero constant value unless the cart is constantly accelerating, which would indicate an unstable system. Finally, if P doesn't have a pole or zero at $s = 0$, we can solve the tracking problem by taking C of the form

$$C(s) = \frac{1}{s} C_1(s)$$

and designing C_1 to stabilize the feedback loop. For example we could design an observer-based controller $C_1(s)$ to stabilize $P(s)/s$. Thus integral control is the key to tracking constant references.

7.2 Distillation Columns

This is a brief description of a multivariable control regulator problem. It was first written by Professor Jorg Raisch, now of the Technical University of Berlin, and is based on a real experimental setup at Stuttgart University. In what follows, the pronoun "we" refers to Professor Raisch and other Stuttgart researchers.

A binary distillation control problem

The plant

Distillation is one of the most important processes in the chemical industries. Its objective is to separate a mixture of chemical components, in this case two alcohols, methanol and propanol. These boil at different temperatures, so by heating the mixture to a temperature between their boiling points, they can be separated; the one that vaporized has to be condensed back into a liquid. In this case methanol boils at a lower temperature.

The Stuttgart system is shown in Figure 1. It is a staged distillation column of 10 metres height and consists of 40 trays, consecutively numbered from top to bottom, a reboiler, and a condenser. The methanol and propanol mixture is continuously fed (labeled "feed") into the column on tray 22. The purpose of control is to maintain bottom and top product purity despite variations in feed flow and feed concentration of up to $\pm 20\%$. Product purity is defined by the concentration of the high-boiling component (propanol) in the bottom product and the concentration of the light-boiling component (methanol) in the distillate. Both concentrations should be greater than 0.999.

Choice of sensors and actuators

Although we ultimately want to control the concentration of propanol and methanol at the top and the bottom of the column, we do not measure them. Concentration measurements are expensive. They also introduce additional time delays into the control loop. Fortunately, as in most binary distillation processes, there is a one-to-one correspondence between temperature and concentrations on each tray. We can therefore control tray temperatures instead of concentrations.

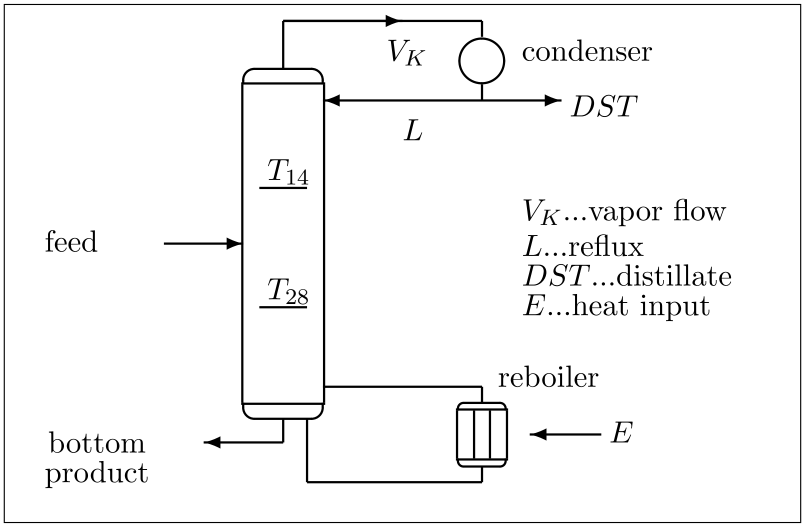


Figure 1: Schematic representation of distillation column

Nonlinear dynamic simulation was used to find a suitable steady-state and to assess the effects of feed disturbances on temperatures on different trays. The simulation model is based on mass balances, energy balances, and thermodynamical correlations on each tray. It consists of a set of 320 nonlinear differential and algebraic equations, which are implemented in a simulation package at Stuttgart University. From simulation experiments it was found that the temperatures on trays 14 and 28 (T_{14}, T_{28}) are among the most sensitive ones with respect to feed disturbances. Therefore, temperature sensors were placed on those trays. By keeping T_{14} and T_{28} within a specified range around their steady-state values, specifications for top and bottom product purity can be met. So these two temperatures are the plant outputs to be regulated.

As is often done in binary distillation problems, we chose heat input E (the energy input to the reboiler) and reflux ratio $\epsilon = L/V_K$ as the control inputs.

The Linear Model

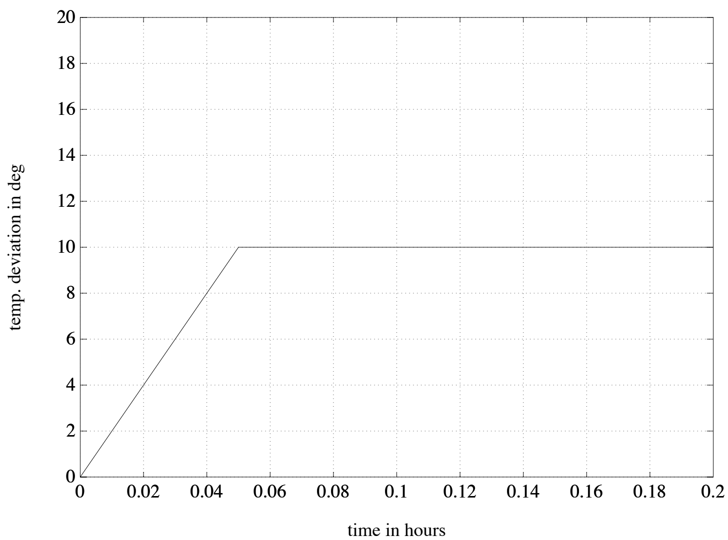
From step responses to E and ϵ, a simple linear input-output plant model was fit:

$$\begin{pmatrix} \Delta T_{14} \\ \Delta T_{28} \end{pmatrix} = G(s) \begin{pmatrix} \Delta\epsilon \\ \Delta E \end{pmatrix} \tag{7.1}$$

The symbol Δ means ‘deviation from set point’. A state-space realization for $G(s)$ was developed, the units being E in kW, temperature in degrees Kelvin, and all time constants in hours. The linearity assumption is quite far from reality, as—for the particular steady-state considered—gain and time constants in each transmission channel can vary by up to 300% depending on the size and sign of the input step. Nevertheless, if the controller is fast enough and confines the plant to a small vicinity of the steady state, the plant model (7.1) and a simple (and rather “tight”) uncertainty model have been found adequate for controller design.

Disturbance model

We don't have an explicit disturbance model, but the following makes sense: A step disturbance of ±20% in both feed flow and feed concentration (of propanol) corresponds (roughly) to a ramp disturbance of the size shown here:



This disturbance is acting on the measured variables, T_{14}, T_{28}; this is referred to as "equivalent output disturbance".

Performance specifications

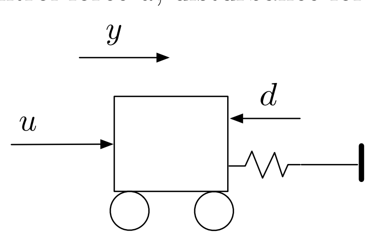
We want

- zero steady-state error
- to reject disturbances fast enough so that we stay within the range of validity of the linear model.

7.3 Problem Setup

Now we turn to the state-space theory of tracking and regulation—the plant output should track a reference signal, such as a step, ramp, or sinusoid, and/or a plant disturbance should be rejected.

Consider a cart/spring with control force u, disturbance force d, and position y:



We want the cart to follow a reference r in spite of the disturbance. To keep things simple, let's take the plant equation to be

$$M\ddot{y} = u - Ky - d.$$

This is an "$f = ma$ equation" where M is the mass and K the spring constant. To make things even simpler, let's take $M = K = 1$:

$$\ddot{y} = u - y - d.$$

Suppose we know that r is a constant (or a step) but we don't know its value; and we know that d is a sinusoid of frequency 10 rad/s but of unknown amplitude and phase. Then we know equations that can generate these signals, namely,

$$\dot{r} = 0, \quad \ddot{d} + 100d = 0.$$

We call these two equations together the **exomodel**, "exo" meaning "from outside the plant". The only unknowns are the initial conditions of these equations. Since we're not saying anything about the magnitudes of these signals, the most we could try to achieve is asymptotic regulation: $r(t) - y(t)$ tends to zero. We want to design a controller to achieve this. We restrict the controller to have input $r - y$ and output u.

We're going to develop a state-space theory for this problem, so it's convenient to make a state model of the plant and exomodel. The setup we want is a plant with state x_1 and an exomodel with state x_2 like this:

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_3x_2 + B_1u \\ \dot{x}_2 &= A_2x_2 \\ e &= D_1x_1 + D_2x_2. \end{aligned}$$

The output e is the signal that we want to go to zero, typically, a tracking error. The exogenous signal x_2 also enters the plant via A_3x_2, a disturbance. It is natural to **assume** that all the eigenvalues of A_2 are unstable (but no assumption is made yet about A_1). For conciseness, the two states can be combined:

$$\dot{x} = Ax + Bu, \quad e = Dx, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

Let us set up the cart example like this. Taking the state variables

$$x_1 = (y, \dot{y}), \quad x_2 = (r, d, \dot{d})$$

and the output $e = r - y$, we have

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -100 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \quad D = \left[\begin{array}{cc|ccc} -1 & 0 & 1 & 0 & 0 \end{array}\right].$$

The partition lines indicate the blocks

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

We see that the eigenvalues of A_2 are $0, \pm 10j$, the frequencies of r and d. So A_2 is completely unstable—no stable eigenvalues.

The **regulator problem** is to design a controller, with input e and output u, such that the feedback loop is stable, meaning the plant state $x_1(t)$ and the controller state go to zero when $x_2(0) = 0$, and the output is regulated, meaning $e(t)$ goes to zero for all initial conditions.

7.4 Tools for the Solution

In this section we develop the tools to solve the regulator problem. The notation is local to this section; for example, there will be an A_1 but it won't be the same as in other sections; however, A_2 will be the same.

Consider the system

$$\dot{x} = Ax, \ e = Dx, \ A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \ D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$$

where A_1 is stable and A_2 has all its eigenvalues in the closed right half-plane. We're interested in when $e(t)$ goes to 0 for every $x(0)$. If $A_3 = 0$, that is, A is block diagonal, the question is easy: $e(t)$ goes to 0 for every $x(0)$ iff $D_2 = 0$. This follows from the equation

$$e(t) = D_1 e^{A_1 t} x_1(0) + D_2 e^{A_2 t} x_2(0).$$

To answer the question when $A_3 \neq 0$, it would be beneficial to transform A so that it becomes block diagonal.

The lower-left block of A being 0 is the sign of an invariant subspace, namely, the subspace of all vectors of the form $x = (x_1, 0)$. That is, if x has this form, so does Ax:

$$x = (x_1, 0) \implies Ax = (A_1 x_1, 0).$$

The invariant subspace can therefore be written as

$$\operatorname{Im} T_1, \quad T_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Note that

$$AT_1 = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} A_1,$$

i.e., $AT_1 = T_1 A_1$. This equation relates to Lemma 3.4.1.

Trying to transform A

$$\text{from} \quad \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

is related to finding another invariant subspace of A. Suppose we could find a matrix T_2 such that $AT_2 = T_2 A_2$ and T_2 has the form

$$T_2 = \begin{bmatrix} X \\ I \end{bmatrix}.$$

Putting T_1 and T_2 together we would have

$$\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

That is, the similarity transformation $T^{-1}AT$, where $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, would block diagonalize A into

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Now the equation

$$AT_2 = T_2 A_2, \quad T_2 = \begin{bmatrix} X \\ I \end{bmatrix}$$

is exactly the same as

$$A_1 X - X A_2 + A_3 = 0.$$

To recap, if this equation has a solution X, then A can be block diagonalized.

Thus, we need this result:

Lemma 7.4.1 *Assume A_1 is stable and A_2 has all its eigenvalues in the closed right half-plane. There exists a unique matrix X satisfying the equation*

$$A_1 X - X A_2 + A_3 = 0. \tag{7.2}$$

Proof If $A_2 = 0$, obviously X exists, namely, $X = -A_1^{-1} A_3$. Likewise, if $A_2 = cI$, with c a positive constant, then the equation is

$$(A_1 - cI)X + A_3 = 0.$$

This has a unique solution because c is not an eigenvalue of A_1. The proof in the general case is a bit involved and is therefore omitted. □

Using this X, we can block-diagonalize A by a similarity transformation:

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \quad T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Under the same transformation, D becomes

$$DT = \begin{bmatrix} D_1 & D_1 X + D_2 \end{bmatrix}.$$

Thus, $e(t)$ goes to 0 for every $x(0)$ iff $D_1 X + D_2 = 0$.

Let's summarize:

Lemma 7.4.2 *Suppose*

$$\dot{x} = Ax,\ e = Dx,\ A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix},\ D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$$

where A_1 is stable and A_2 has all its eigenvalues in the closed right half-plane. Then $e(t)$ goes to 0 for every $x(0)$ iff $D_1X + D_2 = 0$, where X is the unique solution of

$$A_1X - XA_2 + A_3 = 0.$$

A special case is $A_2 = 0$. This will correspond to the case of constant references and/or disturbances. In this case $X = -A_1^{-1}A_3$. The result is this:

Corollary 7.4.1 *Suppose*

$$\dot{x} = Ax,\ e = Dx,\ A = \begin{bmatrix} A_1 & A_3 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$$

where A_1 is stable. Then $e(t)$ goes to 0 for every $x(0)$ iff $-D_1A_1^{-1}A_3 + D_2 = 0$.

This result is quite intuitive when one notices that $-D_1A_1^{-1}A_3 + D_2$ equals the DC gain matrix from x_2 to e for the system

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_3x_2 \\ e &= D_1x_1 + D_2x_2. \end{aligned}$$

7.5 Regulator Problem Solution

To review, the setup is a plant with state x_1 and an exomodel with state x_2 like this:

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_3x_2 + B_1u \\ \dot{x}_2 &= A_2x_2 \\ e &= D_1x_1 + D_2x_2. \end{aligned}$$

The two states can be combined:

$$\dot{x} = Ax + Bu, \quad e = Dx, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

We **assume** that all the eigenvalues of A_2 are unstable (but no assumption is made yet about A_1). It is also natural to **assume** that (D, A) is observable because we're going to use an observer and we're going to want to have no restriction on where to place poles.

The solution is in two parts. We first look for a state feedback controller, $u = Fx$. Then we implement it via $u = F\hat{x}$ where $\hat{x}$ is from an observer with input e.

So let $u = Fx$. Then the controlled system is

$$\dot{x} = (A + BF)x, \quad e = Dx,$$

where

$$A + BF = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} A_1 + B_1F_1 & A_3 + B_1F_2 \\ 0 & A_2 \end{bmatrix}.$$

Clearly feedback stability is equivalent to stability of $A_1 + B_1F_1$. Then asymptotic regulation is equivalent to the condition that $e(t)$ goes to 0 for every $x(0)$.

Theorem 7.5.1 *Assume A_2 has only unstable eigenvalues. Then the regulator problem is solvable by some $u = Fx$ iff (A_1, B_1) is stabilizable and there exist matrices X, U such that*

$$A_1X - XA_2 + A_3 + B_1U = 0, \quad D_1X + D_2 = 0. \tag{7.3}$$

Proof Necessity. Assume $u = Fx$ solves the regulator problem. Certainly (A_1, B_1) is stabilizable. By Lemma 7.4.2, asymptotic regulation implies there exists a matrix X such that

$$(A_1 + B_1F_1)X - XA_2 + A_3 + B_1F_2 = 0, \quad D_1X + D_2 = 0. \tag{7.4}$$

These can be written

$$A_1X - XA_2 + A_3 + B_1U = 0, \quad D_1X + D_2 = 0,$$

where $U = F_1X + F_2$.

Sufficiency. Choose F_1 so that $A_1 + B_1F_1$ is stable. Solve (7.3) for X, U and set $F_2 = U - F_1X$. Then (7.4) holds, so asymptotic regulation follows from Lemma 7.4.2. □

Let us look at the solvability condition for the case $A_2 = 0$, constant exogenous signals. Equation (7.3) becomes

$$A_1X + A_3 + B_1U = 0, \quad D_1X + D_2 = 0,$$

that is,

$$\begin{bmatrix} A_1 & B_1 \\ D_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} + \begin{bmatrix} A_3 \\ D_2 \end{bmatrix} = 0.$$

This is a linear matrix equation in X, U and it has a solution iff

$$\text{rank} \begin{bmatrix} A_1 & B_1 & A_3 \\ D_1 & 0 & D_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 & B_1 \\ D_1 & 0 \end{bmatrix}.$$

If the plant is square, that is, $\dim u = \dim e$, then the matrix

$$\begin{bmatrix} A_1 & B_1 \\ D_1 & 0 \end{bmatrix}$$

is square, and a sufficient condition for solvability of (7.3) is that this matrix is invertible.

Now we turn to designing a controller with input e, not x. Assuming (D, A) is observable, we can select L so that $A + LD$ is stable. The full-state observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(D\hat{x} - e).$$

Setting $u = F\hat{x}$, we get the observer-based controller

$$\dot{\hat{x}} = (A + BF + LD)\hat{x} - Le, \quad u = F\hat{x}.$$

Thus the controller transfer function is

$$C(s) = \left[\begin{array}{c|c} A + BF + LD & -L \\ \hline F & 0 \end{array}\right].$$

Theorem 7.5.2 *Assume (D, A) is observable and A_2 has only unstable eigenvalues. Then the regulator problem is solved by the observer-based controller if $u = Fx$ is a state-feedback solution.*

Instead of the proof, let's do the cart example from start to finish.

Example We start with

$$A = \left[\begin{array}{rr|rrr} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -100 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \quad D = \left[\begin{array}{rr|rrr} -1 & 0 & 1 & 0 & 0 \end{array}\right].$$

We check that A_2 has no stable eigenvalues, that (D, A) is observable, and that (A_1, B_1) is stabilizable, in fact controllable.

Next, we select F_1 to stabilize $A_1 + B_1F_1$. Arbitrarily selecting the eigenvalues to be -1, we get

$$F_1 = \left[\begin{array}{cc} 0 & -2 \end{array}\right].$$

Next, we have to check solvability of

$$A_1X - XA_2 + A_3 + B_1U = 0, \quad D_1X + D_2 = 0.$$

for X, U. The easiest way to do this is to try to solve them. So write

$$X = \left[\begin{array}{ccc} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{array}\right], \quad U = \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]$$

and substitute them into the equations. You will get 9 equations in the 9 unknowns. These indeed have a unique solution:

$$X = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad U = \left[\begin{array}{ccc} 1 & 1 & 0 \end{array}\right].$$

Then

$$F_2 = U - F_1X = \left[\begin{array}{ccc} 1 & 1 & 0 \end{array}\right].$$

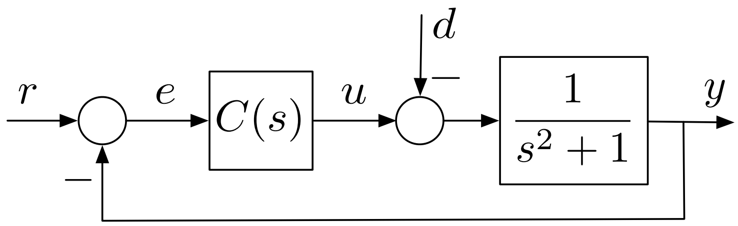
Finally, assigning the eigenvalues of $A + LD$ at -1, we get

$$L = \left[\begin{array}{ccccc} 5 & -91 & -0.01 & 490 & -9005 \end{array}\right]^T.$$

The resulting controller transfer function

$$C(s) = \left[\begin{array}{c|c} A+BF+LD & -L \\ \hline F & 0 \end{array}\right] . = \frac{-672s^4 + 8015s^3 - 679s^2 + 8007s + 1}{s(s^4 + 7s^3 + 20s^2 + 700s - 8000)}.$$

The controller has poles at $0, \pm 10j$, as it must to track the step and reject the disturbance. The structure has the familiar block diagram:



$\square$

7.6 Unobservability

What happens when (D, A) is not observable and we can't construct an observer with input e and arbitrary pole locations? Here are some examples to illustrate.

Example Consider the cart with neither spring nor disturbance:

$$\ddot{y} = u$$

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right], \quad D = \left[\begin{array}{cc|c} -1 & 0 & 1 \end{array}\right].$$

Here (D, A) is not observable. The reason is that the setup is redundant: Since the plant is a double integrator, **step-tracking will automatically follow from feedback stability**. So modeling r is unnecessary. In fact, we should not have modeled r, but since we did model it, the model needs pruning.

The precise way to prune the model is to take out the unobservable part. We learned how to do this in the Observability chapter. The unobservable subspace of (D, A) is spanned by the single vector $(1, 0, 1)$. Make this the third column of a transformation matrix W:

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Take the first two columns to be the standard basis vectors. Then $W^{-1}AW, W^{-1}B, DW$ are

$$\tilde{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right], \quad \tilde{D} = \left[\begin{array}{cc|c} -1 & 0 & 0 \end{array}\right].$$

The unobservable part is the third state, which in the new basis is the exomodel state. Pruning out that state, we have the reduced model

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

In this model there is no exomodel and the regulator problem is just a stabilization problem. We select $\bar{F}$ to stabilize $\bar{A} + \bar{B}\bar{F}$. For the eigenvalues to be -1, we get

$$\bar{F} = \begin{bmatrix} -1 & -2 \end{bmatrix}.$$

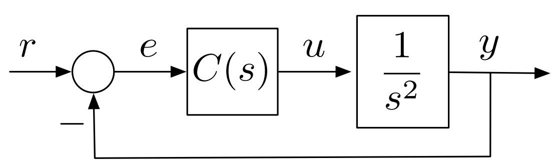
Finally, again assigning the eigenvalues of $\bar{A} + \bar{L}\bar{D}$ at -1, we get

$$\bar{L} = \begin{bmatrix} 2 & 1 \end{bmatrix}^T.$$

The resulting controller from e to u is

$$C(s) = \frac{4s+1}{s^2+4s+6}.$$

The block diagram is



□

The general idea is that the plant model would always itself be controllable, observable, so a lack of observability in (D, A) comes from redundancy in the exomodel. So the pruning is to get rid of that redundancy while preserving the plant. Therefore, the structure of W should be like this: Let's say the plant is dimension n_1 and the unobservable subspace of (D, A) is dimension q. Then the first n_1 columns of W should be the first n_1 columns of the identity matrix (this preserves A_1 and keeps the lower-left block of A to be zero) and the last q columns of W should be a basis for the unobservable subspace.

Example Consider the cart without the spring but subject to the disturbance:

$$\ddot{y} = u - d.$$

We have

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -100 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \quad D = \left[\begin{array}{cc|ccc} -1 & 0 & 1 & 0 & 0 \end{array}\right].$$

We check that A_2 has no stable eigenvalues, that (D, A) is unobservable, and that (A_1, B_1) is controllable. The unobservable subspace of (D, A) is spanned by the single vector $(1, 0, 1, 0, 0)$.

Make this the fifth column of a transformation matrix W, taking the first four columns to be the standard basis vectors. Then compute $W^{-1}AW, W^{-1}B, DW$. You will see that the fifth state variable is unobservable, so prune it from the model. The result is

$$\bar{A} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -100 & 0 \end{array}\right], \quad \bar{B} = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right], \quad \bar{D} = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \end{array}\right].$$

So what we've done is prune out the part of the exomodel corresponding to r, keeping the part corresponding to d. Proceeding as before we get

$$C(s) = \frac{-580s^3 + 8515s^2 + 6s + 1}{s^4 + 6s^3 + 15s^2 + 600s - 8500}.$$

□

Example This plant is a pure integrator:

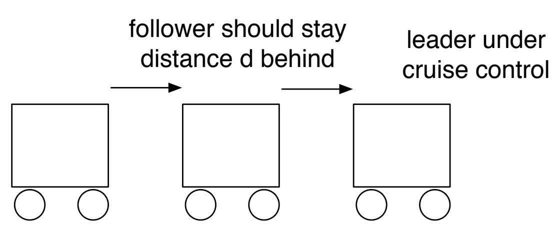
$$\dot{y} = u.$$

Suppose the goal is to have y track a ramp r. The exomodel is a double integrator, so the plant already has "half" an internal model. So there's a redundancy and we need somehow to prune the exomodel down to a single integrator. The setup is

$$A = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right], \quad D = \left[\begin{array}{c|cc} -1 & 1 & 0 \end{array}\right].$$

The rest is left as an exercise. □

Example This continues the platoon example of Chapter 1. We want carts to move in a straight line like this: A designated leader should go at constant speed: the second should follow at a fixed distance d; the third should follow the second at the distance d; and so on.



Let's consider the control problem of the second vehicle, the follower. Denote its position by x_1 and its velocity by u. Then

$$\dot{x}_1 = u.$$

Let the position and velocity of the leader be y_l and $\dot{y}_l$. Then the goal is to have

$$y_l - x_1 = d.$$

So we define the error,

$$e = d - (y_l - x_1).$$

To formulate a problem we can solve, we'll consider the situation where the leader is going at constant speed, i.e., $\ddot{y}_l = 0$. Thus the exomodel has

$$\dot{d} = 0, \quad \ddot{y}_l = 0.$$

Notice that we're modeling d in the exomodel even though we know its actual value. The state of the exomodel is taken to be

$$x_2 = (d, y_l, \dot{y}_l).$$

So

$$\dot{x}_2 = A_2 x_2, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now combine the two states: $x = (x_1, x_2)$. Then

$$\dot{x} = Ax + Bu, \quad e = Dx,$$

where

$$A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \quad D = \left[\begin{array}{c|ccc} 1 & 1 & -1 & 0 \end{array}\right].$$

The partition lines indicate the blocks

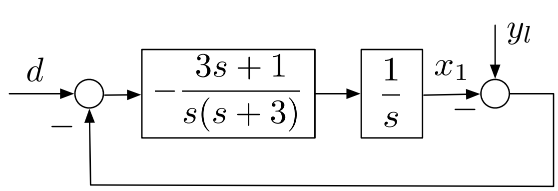
$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

We see that the eigenvalues of A_2 are $0, 0, 0$. So A_2 is completely unstable—no stable eigenvalues. Also, (D, A) is not observable.

See Scilab code next page. The resulting controller, with input e and output u, is

$$C(s) = -\frac{3s+1}{s(s+3)}.$$

The final configuration is



```
// controller for a follower in a platoon.

//initial model


A=[0 0 0 0;0 0 0 0;0 0 0 1;0 0 0 0];
D=[1 1 -1 0];
B=[1 0 0 0]';

// unobservable subspace is spanned by (0,1,1,0), (1,0,1,0)
// transform to At etc (A tilde)

T=[1 0 0 1;0 0 1 0;0 0 1 1;0 1 0 0];
At=inv(T)*A*T;
Dt=D*T;
Bt=inv(T)*B;

// new,reduced model

A1=At(1,1);
A3=At(1,2);
A2=At(2,2);
D1=Dt(1,1);
D2=Dt(1,2);
B1=Bt(1);

// stabilize A1 + B1 F1

F1=-ppol(A1,B1,-1);

// solve for F2

F2=1;

// observer

A=[A1 A3;0*A2 A2];
D=[D1 D2];
F=[F1 F2];
B=[B1;0*B1];

L=-ppol(A',D',[-1 -1]);
L=L';

// controller state matrices

Ac=A+B*F+L*D;
Bc=-L;
Cc=F;

// controller transfer function

[Con]=syslin('c',Ac,Bc,Cc);
[C]=ss2tf(Con);
```

There is a general theory about pruning the unobservable part of (D, A) in the regulator problem.

In the interests of time we'll skip it. For us, we'll use the pruning procedure to get a controller and then check afterwards if it solves the regulator problem.

7.7 More Examples

We'll do additional examples to illustrate some things that can happen, such as non-solvability of the problem.

Example Consider the cart/spring with no disturbance:

$$\ddot{y} = u - y.$$

This is a pure step-tracking problem with

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right], \quad D = \left[\begin{array}{cc|c} -1 & 0 & 1 \end{array}\right].$$

We check that A_2 has no stable eigenvalues—in fact $A_2 = 0$—that (D, A) is observable, and that (A_1, B_1) is controllable.

Next, we select F_1 to stabilize $A_1 + B_1F_1$. For the eigenvalues to be -1, we get the same F_1,

$$F_1 = \left[\begin{array}{cc} 0 & -2 \end{array}\right].$$

Next, we have to check solvability of

$$A_1X + A_3 + B_1U = 0, \quad D_1X + D_2 = 0.$$

for X, U. We can write these as

$$\left[\begin{array}{cc} A_1 & B_1 \\ D_1 & 0 \end{array}\right] \left[\begin{array}{c} X \\ U \end{array}\right] = -\left[\begin{array}{c} A_3 \\ D_2 \end{array}\right].$$

The matrix on the left is invertible, so we get

$$\left[\begin{array}{c} X \\ U \end{array}\right] = -\left[\begin{array}{cc} A_1 & B_1 \\ D_1 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} A_3 \\ D_2 \end{array}\right].$$

This yields

$$X = \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \quad U = 1.$$

Then $F_2 = U - F_1X = 1$.

Finally, again assigning the eigenvalues of $A + LD$ at -1, we get

$$L = \left[\begin{array}{ccc} 2 & 2 & -1 \end{array}\right]^T.$$

The resulting controller is

$$C(s) = \frac{5s^2 - 4s + 1}{s(s^2 + 5s + 9)}.$$

□

Example Now, an example plant where step-tracking is not feasible. We just need the plant to have a zero at $s = 0$:

$$\ddot{y} = \dot{u} - y$$

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right], \quad D = \left[\begin{array}{cc|c} 0 & -1 & 1 \end{array}\right].$$

Again, $A_2 = 0$, (D, A) is observable, and (A_1, B_1) is controllable. We have to check solvability of

$$\begin{bmatrix} A_1 & B_1 \\ D_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = -\begin{bmatrix} A_3 \\ D_2 \end{bmatrix}.$$

The numbers are

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} X \\ U \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This isn't solvable. □

The situation may arise where the reference value r is actually known. For example, for the maglev problem we had pre-specified that we wanted to regulate the ball exactly at 1 cm. Then the problem is really one of stabilization about a nonzero equilibrium point, as illustrated by an example.

Example The cart/spring system

$$\ddot{y} = u - y$$

where the desired position is $y = r = 1$, a fixed known value. Of course, we could use regulator theory, but we don't need to. The model is

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We want to stabilize the system at

$$x = (1, 0).$$

Consider the control law

$$u = Fx + \overline{u},$$

where $\overline{u}$ is a constant to be determined. Then

$$\dot{x} = (A + BF)x + B\overline{u}.$$

If $A+BF$ is stable, then $x(t)$ converges to $-(A+BF)^{-1}B\overline{u}$. Let's take

$$F = \begin{bmatrix} 0 & -2 \end{bmatrix}.$$

Then the solution of

$$-(A+BF)^{-1}B\overline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

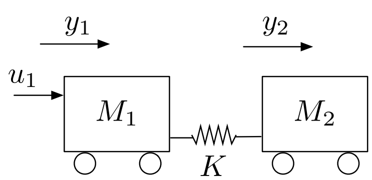
is $\overline{u} = 1$. If only y, not the full state, is sensed, an observer-based controller can be used:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ u &= F\hat{x} + \overline{u}. \end{aligned}$$

Again, $y(t)$ converges to the desired value. □

7.8 Problems

1. Consider the 2-cart system



   Take $M_1 = M_2 = 1$, $K = 2$. The control input is u_1 and we want y_1 to follow a step r. Solve the regulator problem.

2. Solve the regulator problem for the following cases:

plant	spec	r	d
$\dot{y} = y + u$	$r - y \longrightarrow 0$	ramp	
$\dot{y} = u$	$r - y \longrightarrow 0$	ramp	
$\dot{y} = y + d + u$	$r - y \longrightarrow 0$	ramp	step
$\dot{y} = d + u$	$y \longrightarrow 0$		sinusoid of freq. 2 rad/s
$\dot{y} = d + u$	$r - y \longrightarrow 0$	ramp	sinusoid of freq. 2 rad/s
$\dot{y} = -y + d + u$	$r - y \longrightarrow 0$	ramp	sinusoid of freq. 2 rad/s

3. Consider the plant

$$\dot{x}_1 = A_1 x_1 + B_1 u, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   The goal is to get $x_1(t)$ to track asymptotically a point $x_2(t)$ moving at unit speed counterclockwise around the unit circle in $\mathbb{R}^2$.

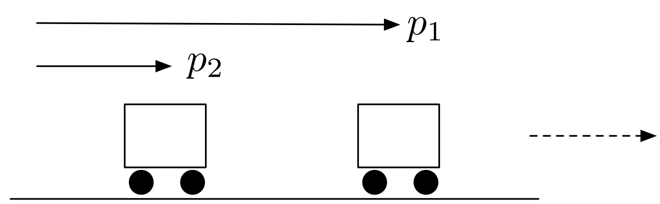
   (a) Find an exomodel $\dot{x}_2 = A_2 x_2$.

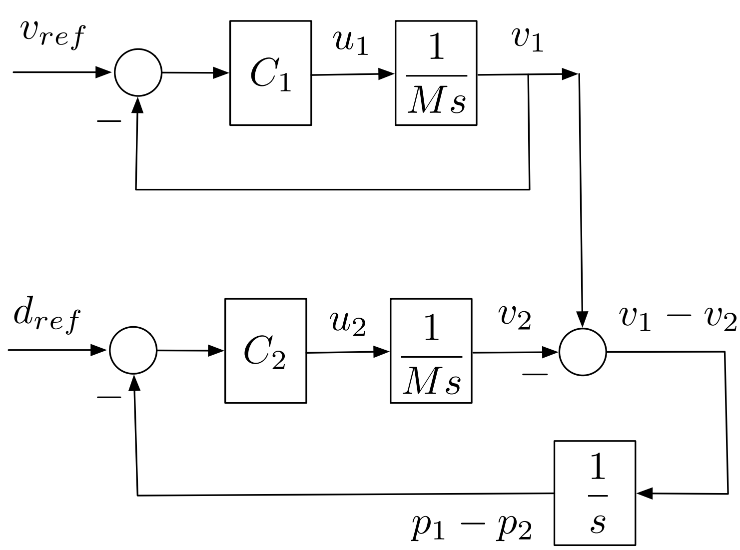
(b) Is the tracking problem solvable?

(c) Repeat with

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. Two cars, 1 and 2; car 1 is the leader and car 2 the follower. Car 1 wants to move at constant speed v_{ref}, while car 2 wants to keep a distance d_{ref} behind. Here's the schematic and the block diagram:





Car 1 has the model $M\dot{v}_1 = u_1$; likewise for car 2. Controller C_1 has input the speed error, while controller C_2 has input the relative position error. The symbols v_i stand for speed and p_i for position.

Take MKS units, take v_{ref} to be the equivalent in m/s of 100 km/hr, take d_{ref} to be 5 m, and take M to be 1500 kgm. Design sensible controllers C_1 and C_2.

5. This model is taken from the Mathworks website. Use Scilab/MATLAB for this problem. The problem relates to the design of a controller for the yaw motion of a Boeing 747 jet transport plane. The plant is fourth order, with state variables sideslip angle, yaw angle, roll rate, and bank angle. There are two inputs, rudder angle and aileron angle. The units are radians for

angles and radians/sec for angle rates. The plant matrices are

$$A_1 = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0729 & 0 \\ -4.75 & 0.00775 \\ 0.153 & 0.1430 \\ 0 & 0 \end{bmatrix}.$$

To get a feel for the open-loop dynamics, simulate and plot yaw angle for an initial value of 10 degrees. Now design a controller so that the yaw angle asymptotically tracks a constant value (step reference). Your controller will first be of the form $u = F_1x_1 + F_2x_2$. Try to place the poles of $A_1 + B_1F_1$ so that when you simulate the yaw angle for a step command of 10 degrees (starting from 0 degrees), the response seems not unreasonable in comparison to the open-loop one. Remember: It's a jumbo jet. Finally, design an observer so that your controller has the yaw angle tracking error as input.