Multi-Agent Systems

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Graph theory: basic concepts

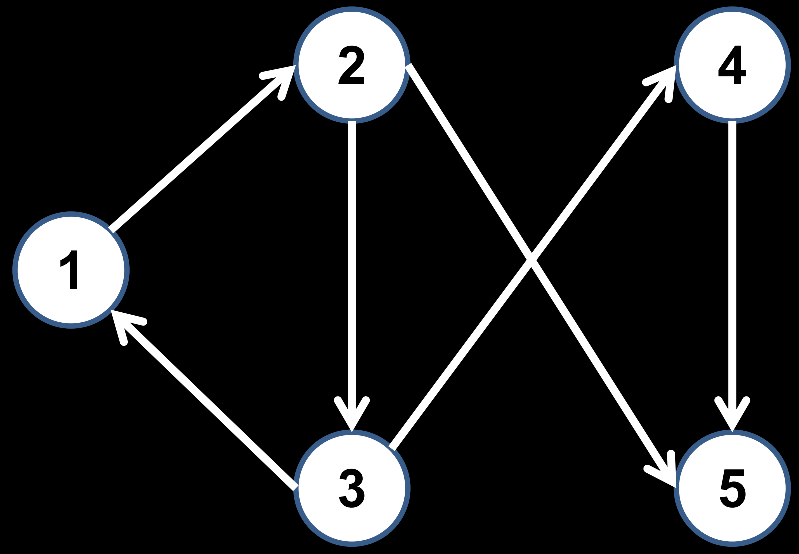
Graph

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node set $\mathcal{V} = \{v_1, \dots, v_n\}$

edge set $\mathcal{E} = \{(v_i, v_j), \dots\}$

example:



Directed, undirected

generally $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is directed (directed graph, or digraph)

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is undirected if $(\forall v_i, v_j \in \mathcal{V})(v_i, v_j) \in \mathcal{E} \Rightarrow (v_j, v_i) \in \mathcal{E}$

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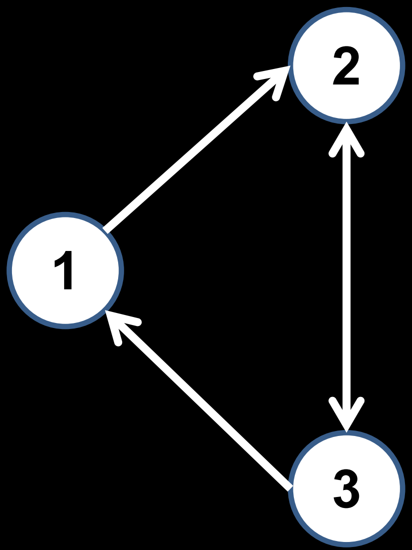

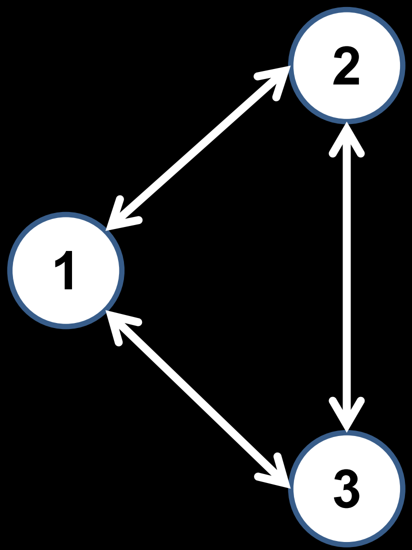

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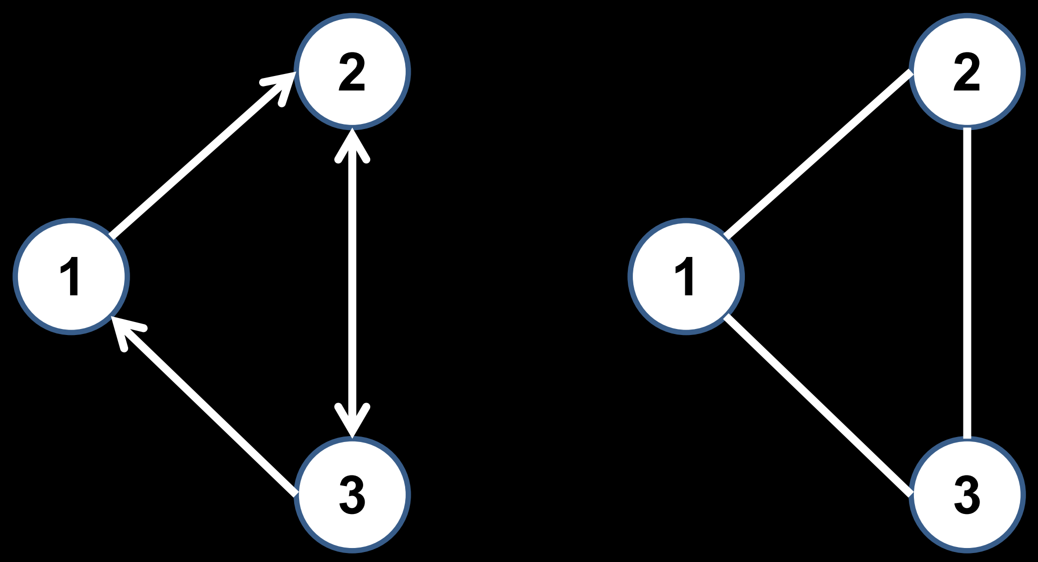
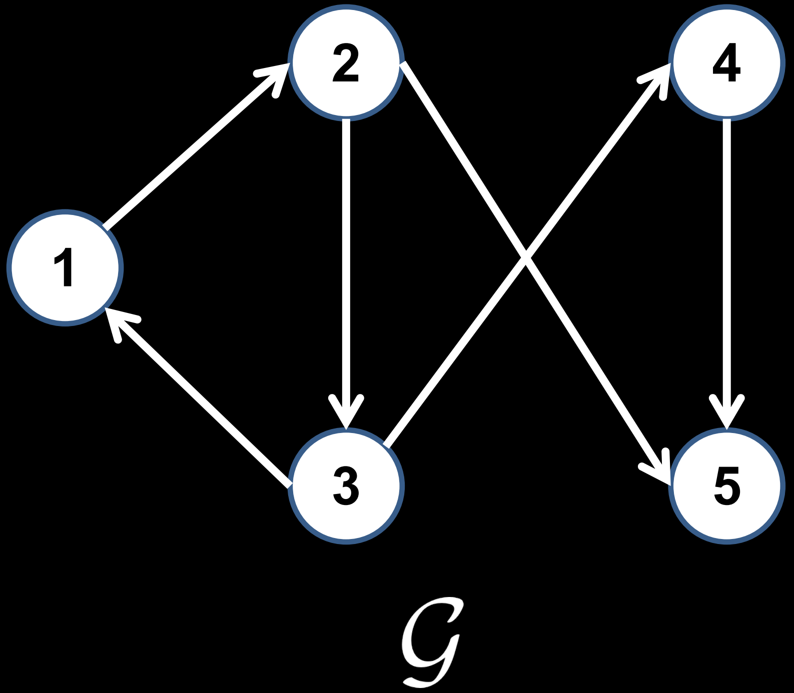
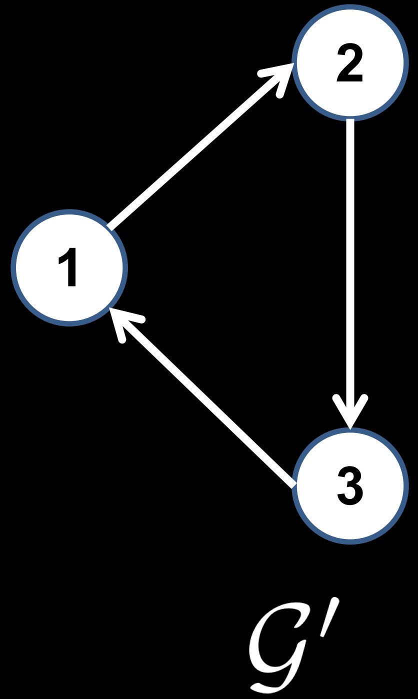


Subgraph

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a subgraph of $\mathcal{G}$
if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$

example:

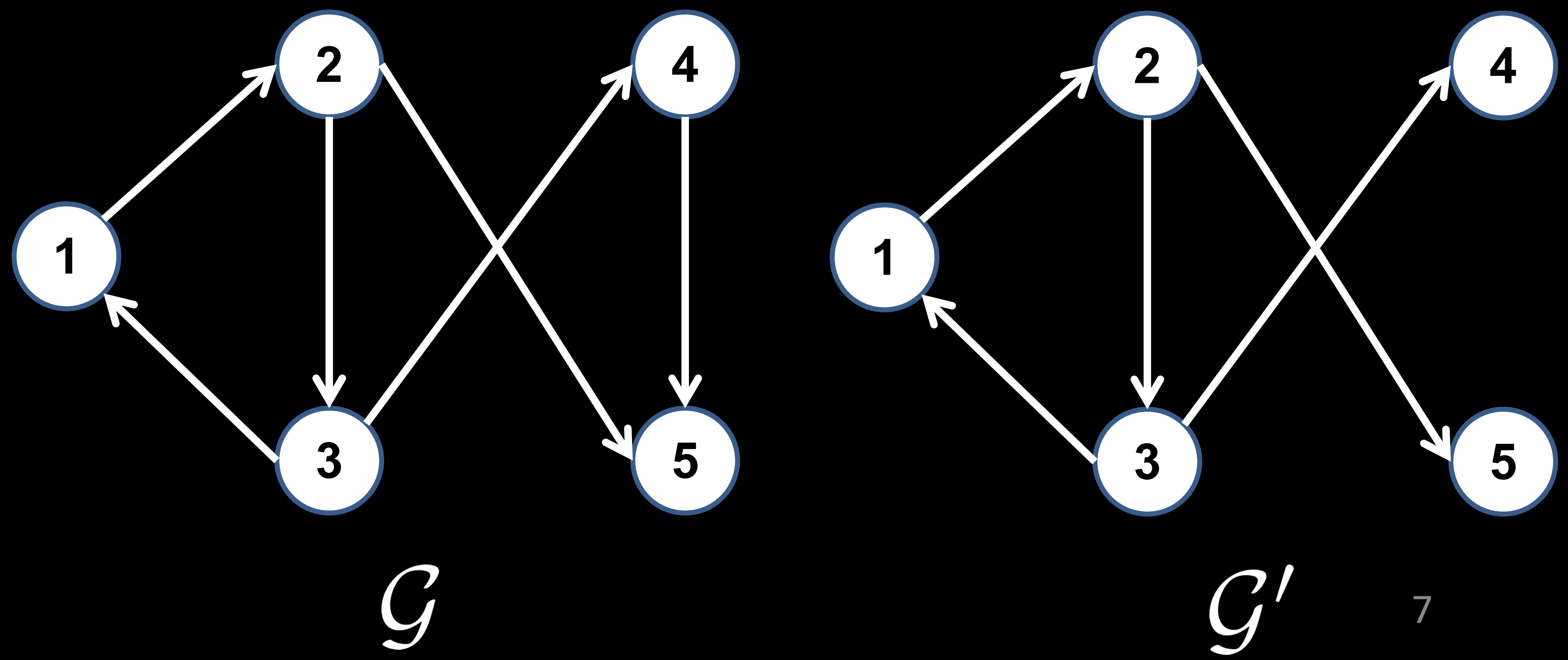




Spanning subgraph

subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
if $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$

example:



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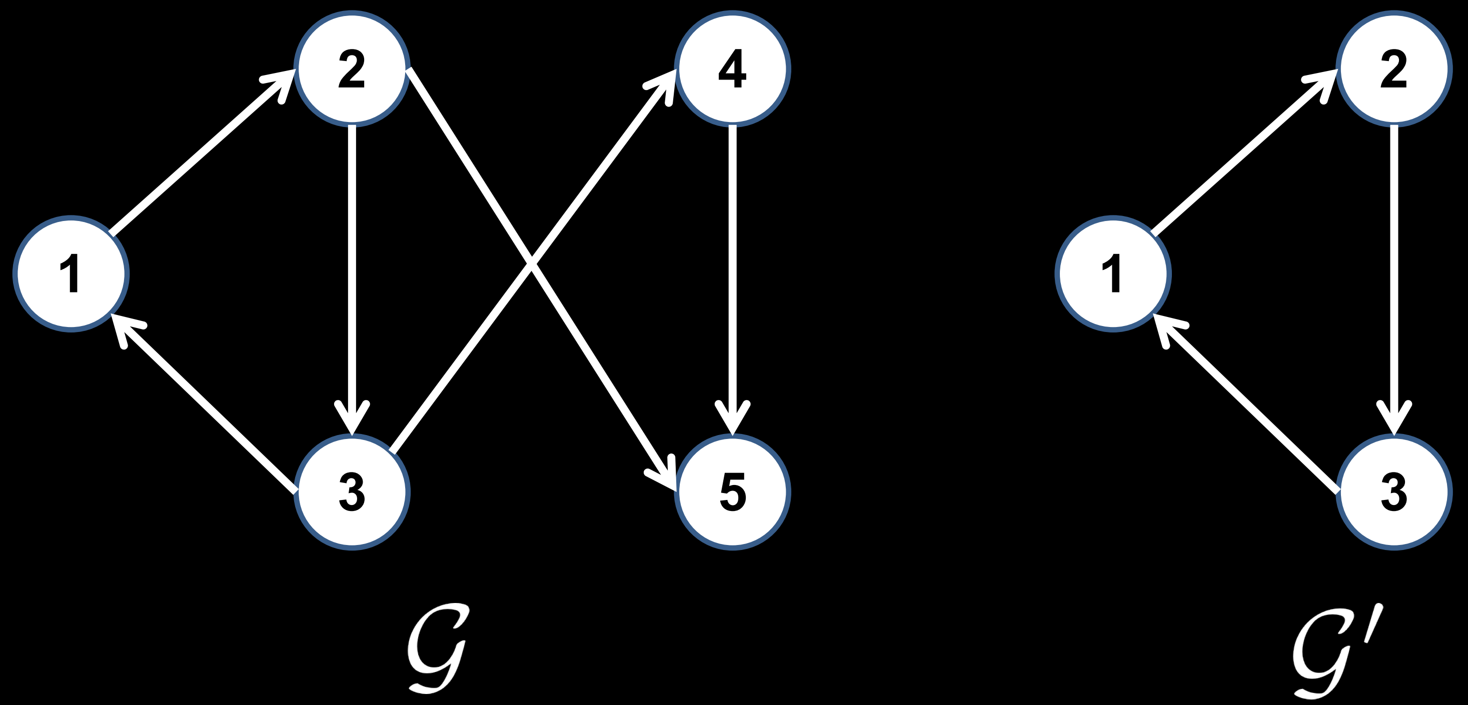
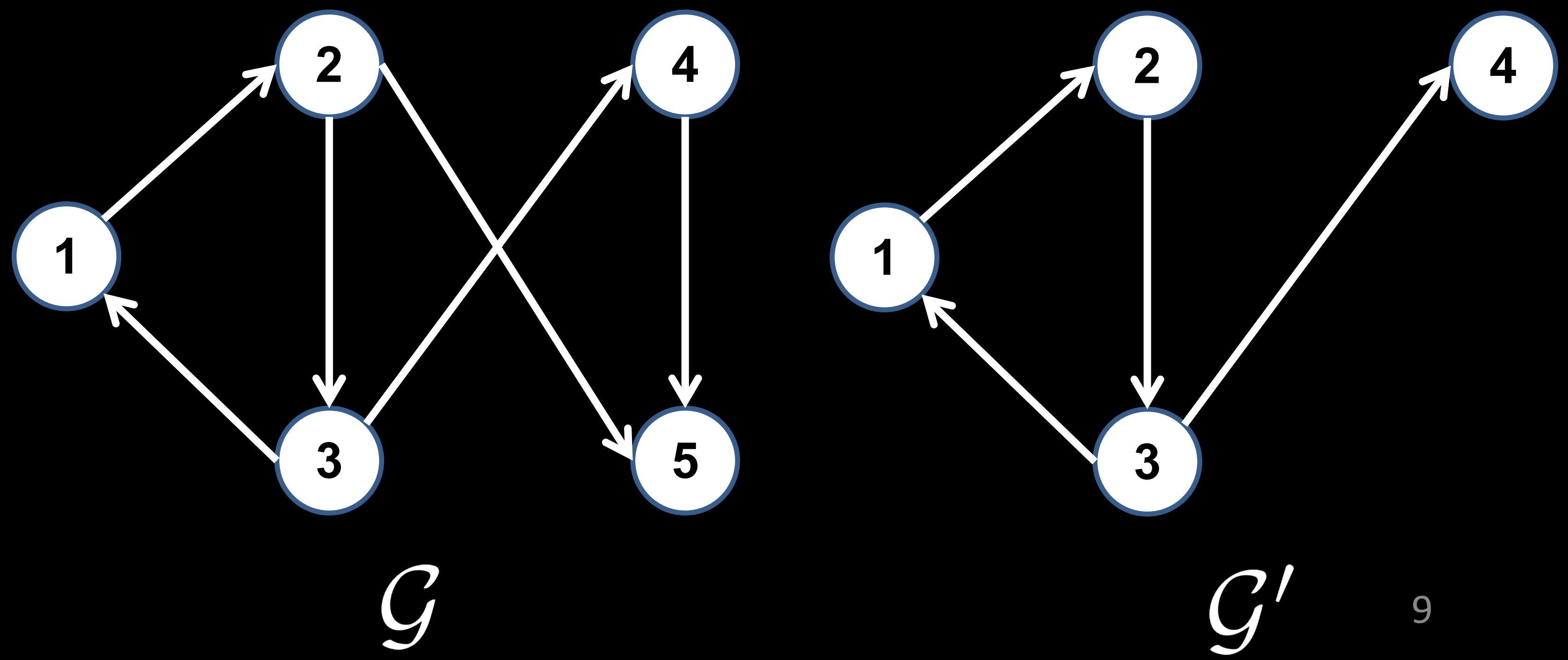


Induced subgraph

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\emptyset \neq \mathcal{V}' \subseteq \mathcal{V}$
induced subgraph by $\mathcal{V}'$ is
$\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}')$

example: $\mathcal{V}' = \{1, 2, 3, 4\}$



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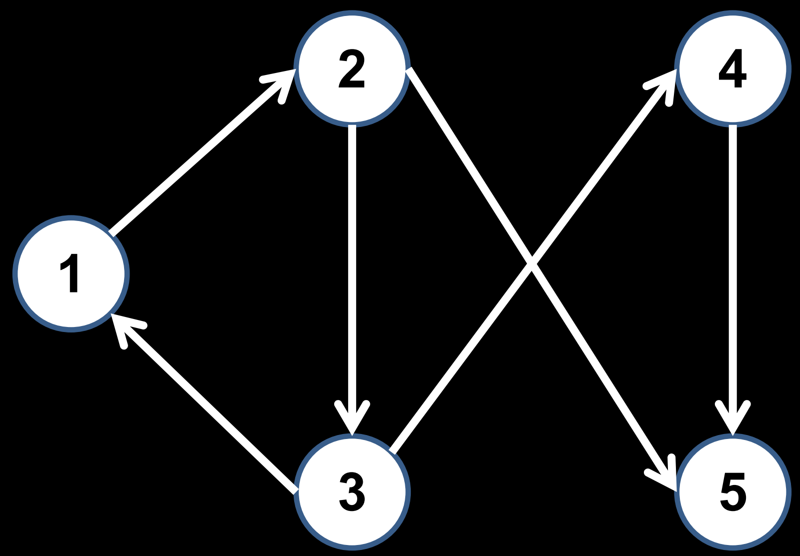
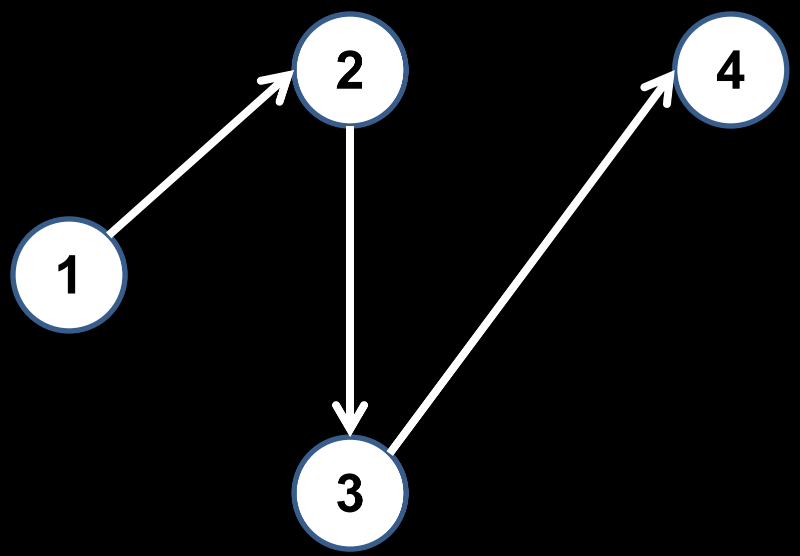


$\mathcal{G}$



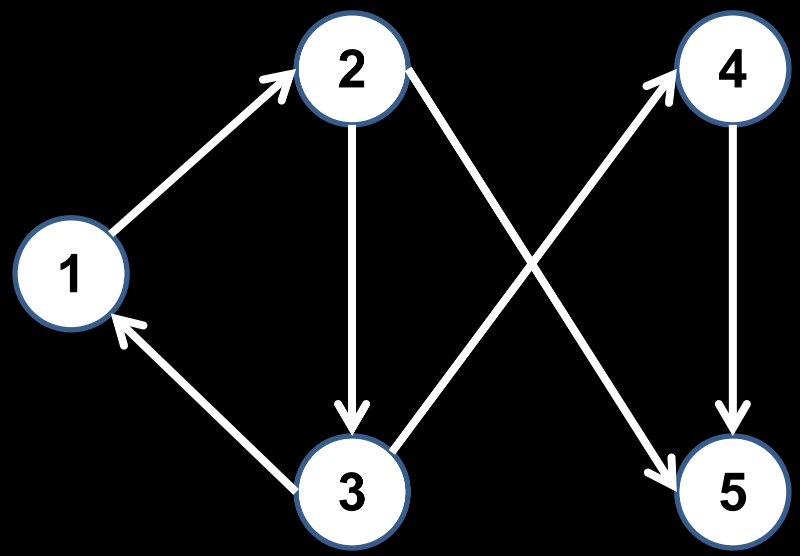
$\mathcal{G}'$

Neighbor

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and node $v \in \mathcal{V}$
neighbor set of v is

$$\mathcal{N}_v = \{u \in \mathcal{V} \mid (u, v) \in \mathcal{E}\}$$

example:



$\mathcal{G}$

$\mathcal{N}_1$
$\mathcal{N}_2$
$\mathcal{N}_3$
$\mathcal{N}_4$
$\mathcal{N}_5$

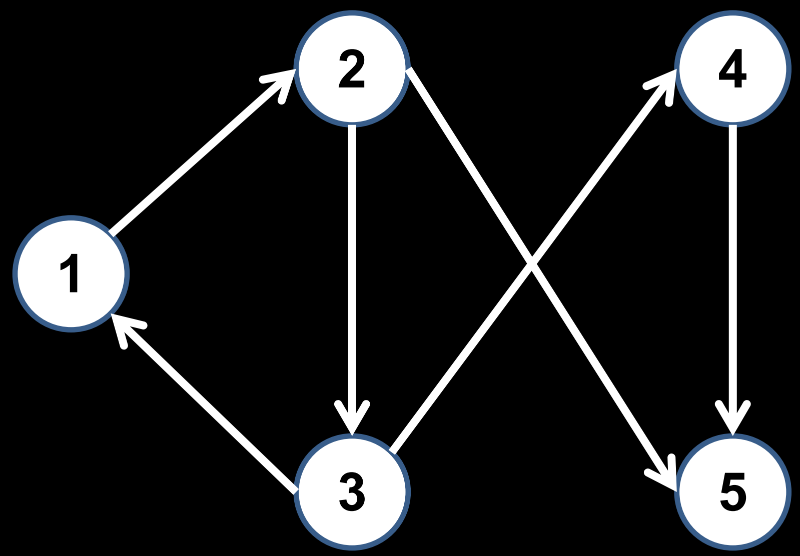
Neighbor

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and node $v \in \mathcal{V}$

out-neighbor set of v is

$$\mathcal{N}_v^o = \{u \in \mathcal{V} \mid (v, u) \in \mathcal{E}\}$$

example:



$\mathcal{G}$

$\mathcal{N}_1^o$

$\mathcal{N}_2^o$

$\mathcal{N}_3^o$

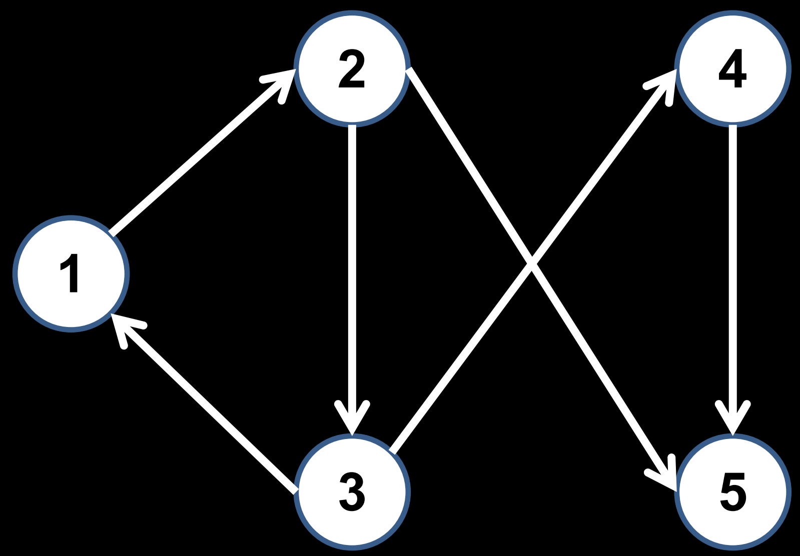
$\mathcal{N}_4^o$

$\mathcal{N}_5^o$

Degree

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and node $v \in \mathcal{V}$
degree of v is $d_v = |\mathcal{N}_v|$
($|\cdot|$: number of elements in the set)

example:



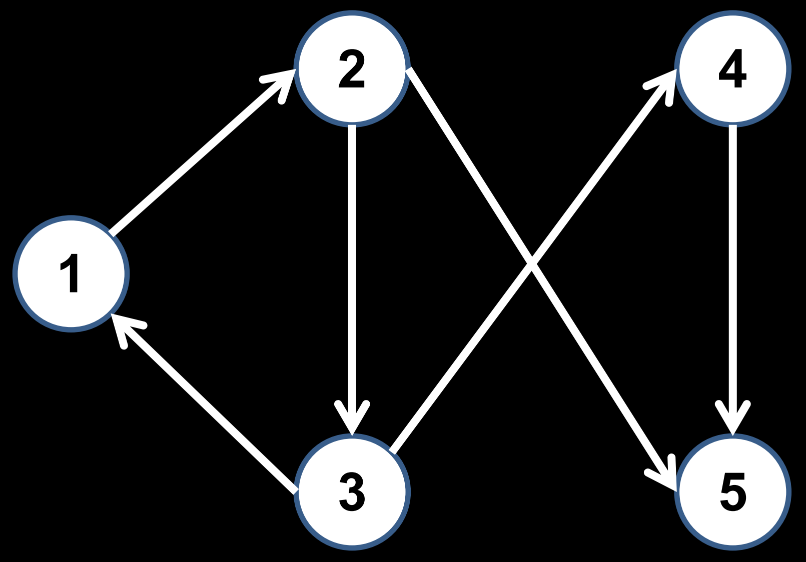
$\mathcal{G}$

Degree

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and node $v \in \mathcal{V}$

out-degree of v is $d_v^o = |\mathcal{N}_v^o|$

example:

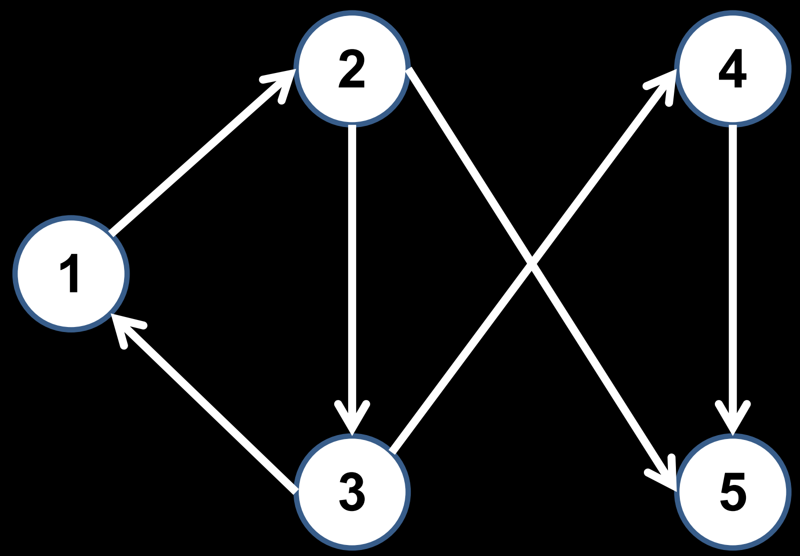


$\mathcal{G}$

Balanced graphs

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
node $v \in \mathcal{V}$ is balanced if $d_v = d_v^o$

example:



$\mathcal{G}$

Balanced graphs

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node $v \in \mathcal{V}$ is balanced if $d_v = d_v^o$

$\mathcal{G}$ is balanced if every v is balanced

(all undirected graphs are balanced)

example: