Multi-Agent Systems

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Last week

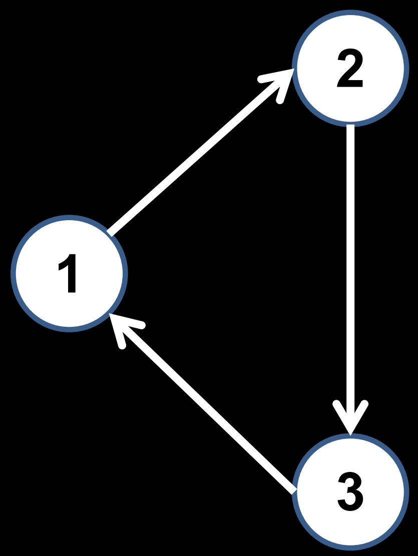
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

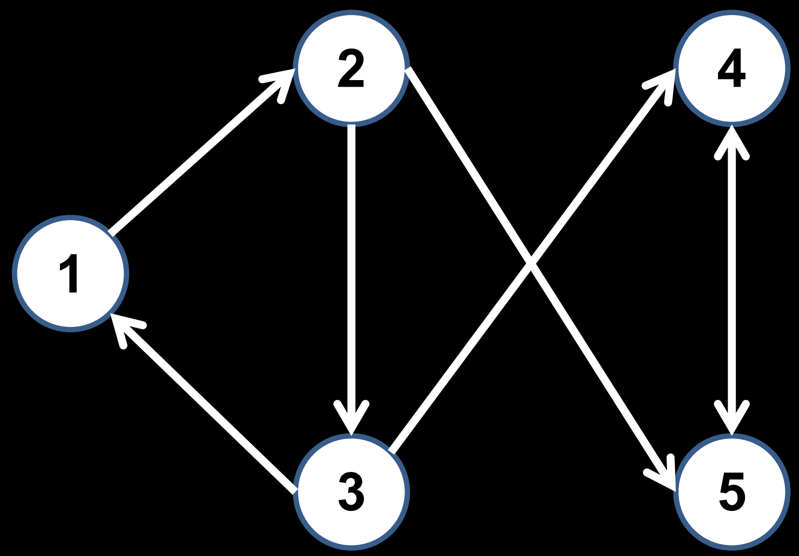
strongly connected

strong component

closed strong component

example:





Last week

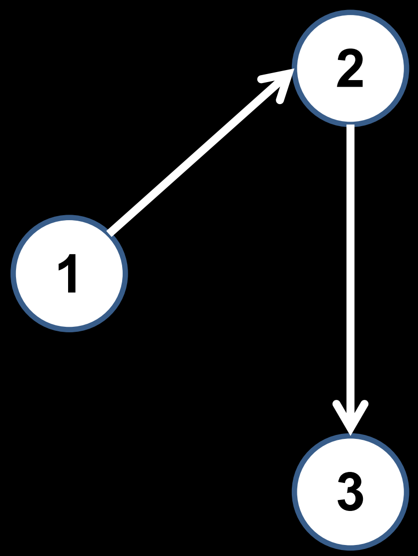
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

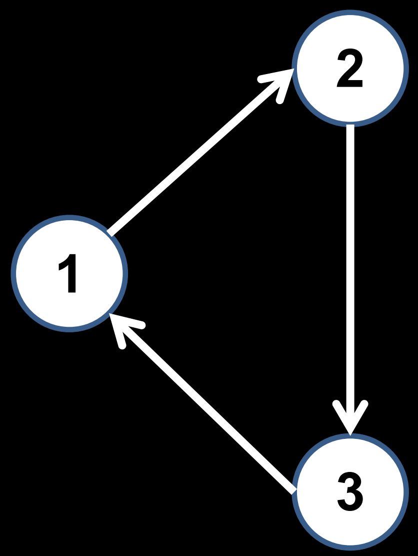
root

spanning tree

$\mathcal{G}$ contains a spanning tree

example:



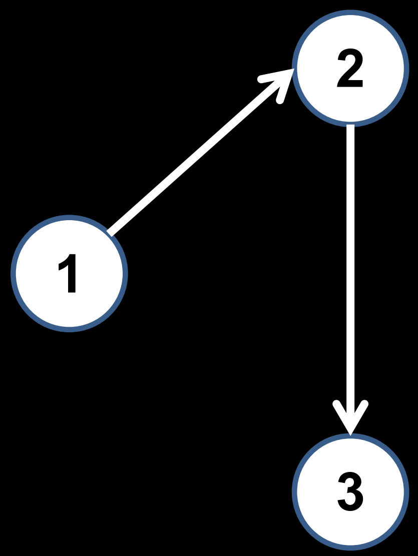


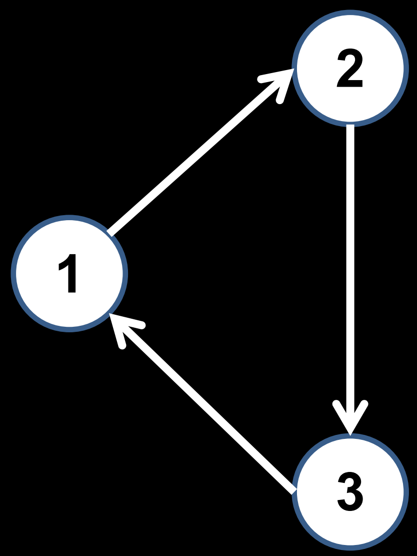
Last week

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$\mathcal{G}$ contains a spanning tree iff $\mathcal{G}$ contains a unique (exactly 1) closed strong component

example:





Last week

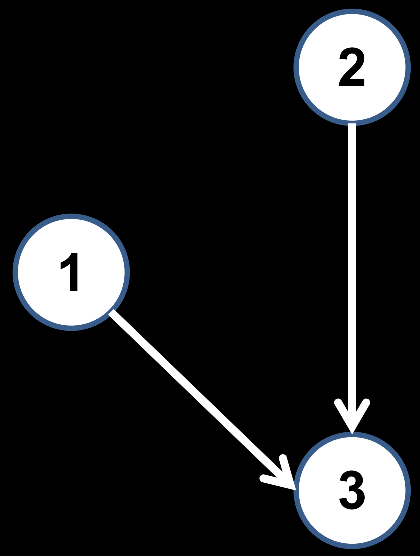
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

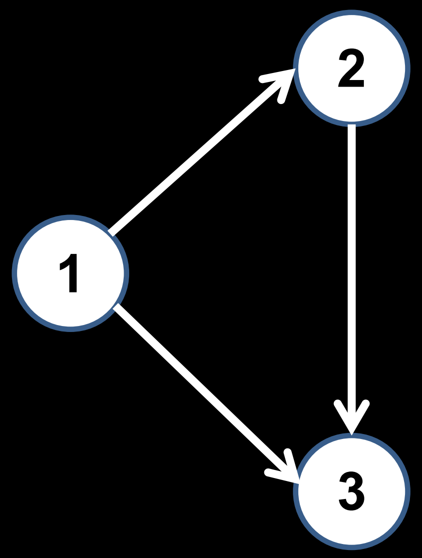
2-root set

spanning 2-tree

$\mathcal{G}$ contains a spanning 2-tree

example:





Fact

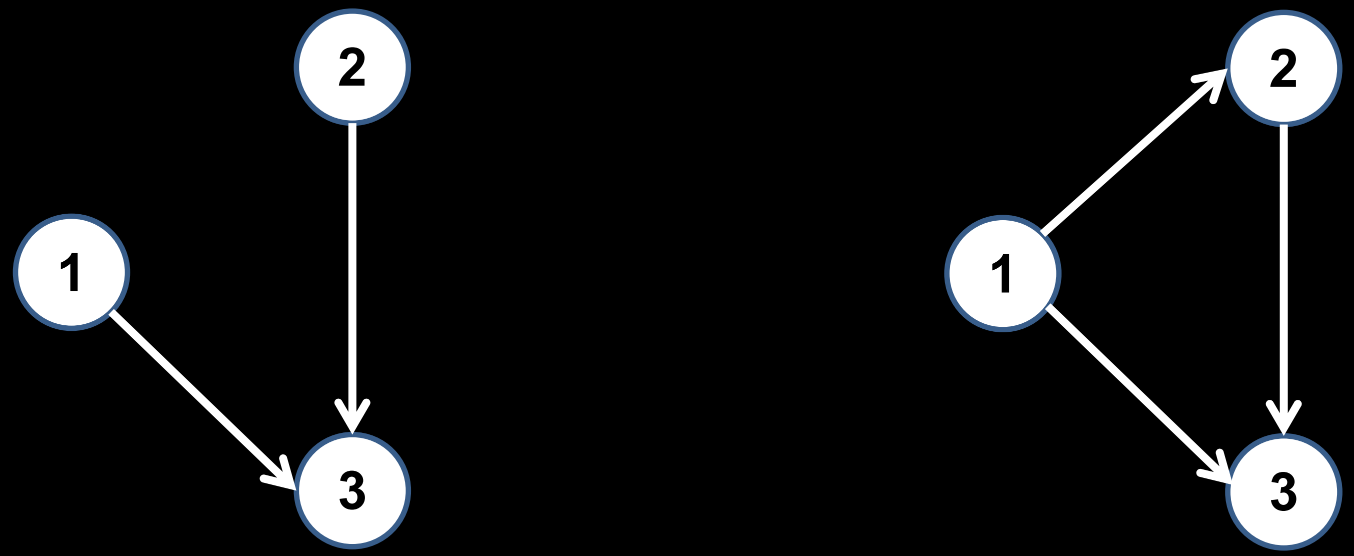
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

if $\mathcal{G}$ contains a spanning 2-tree

then $\mathcal{G}$ contains 1 or 2

closed strong components

example:



Fact

graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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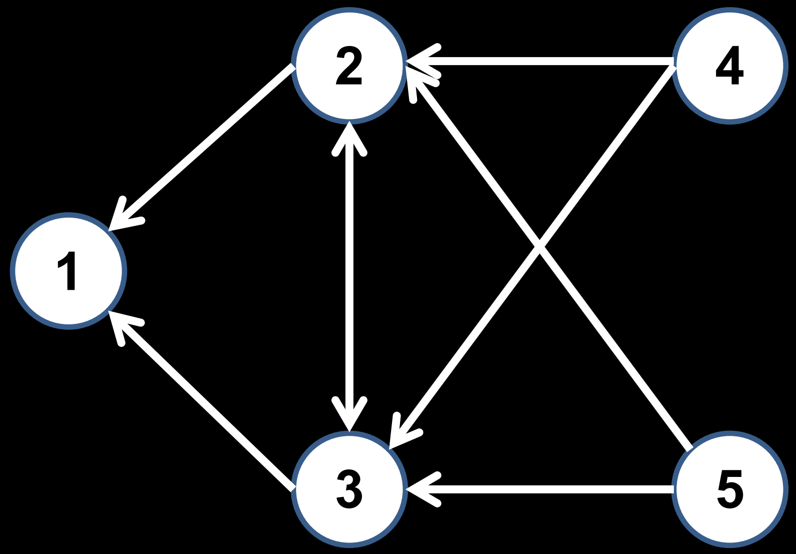
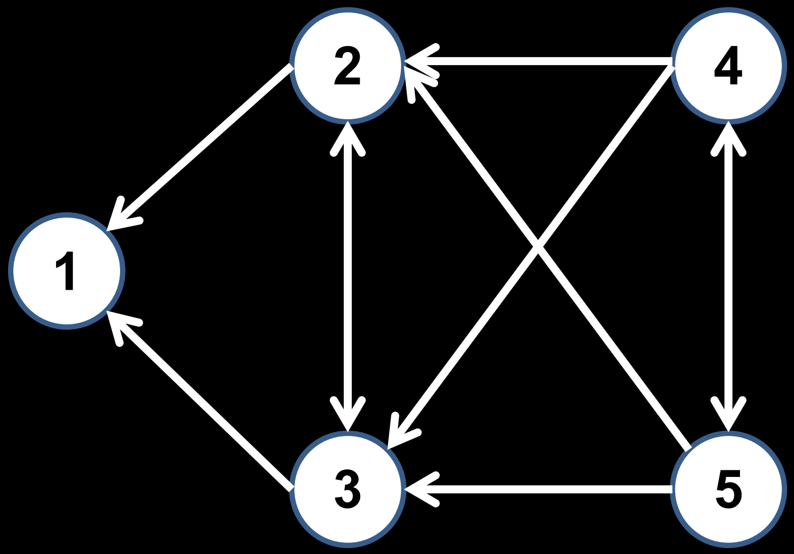
example:





Last week

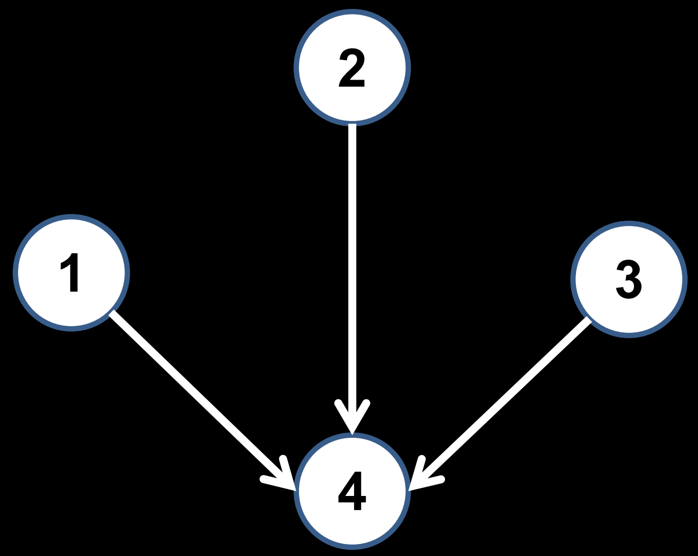
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

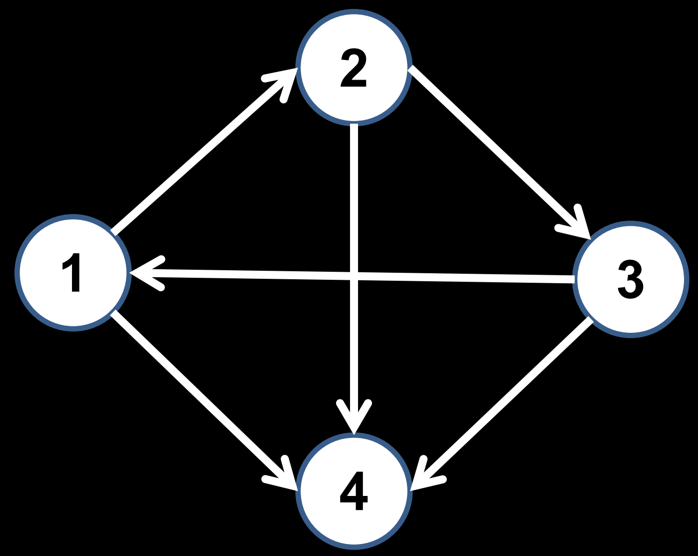
k-root set $(k \geq 2)$

spanning k-tree

$\mathcal{G}$ contains a spanning k-tree

example $(k = 3)$:





Fact

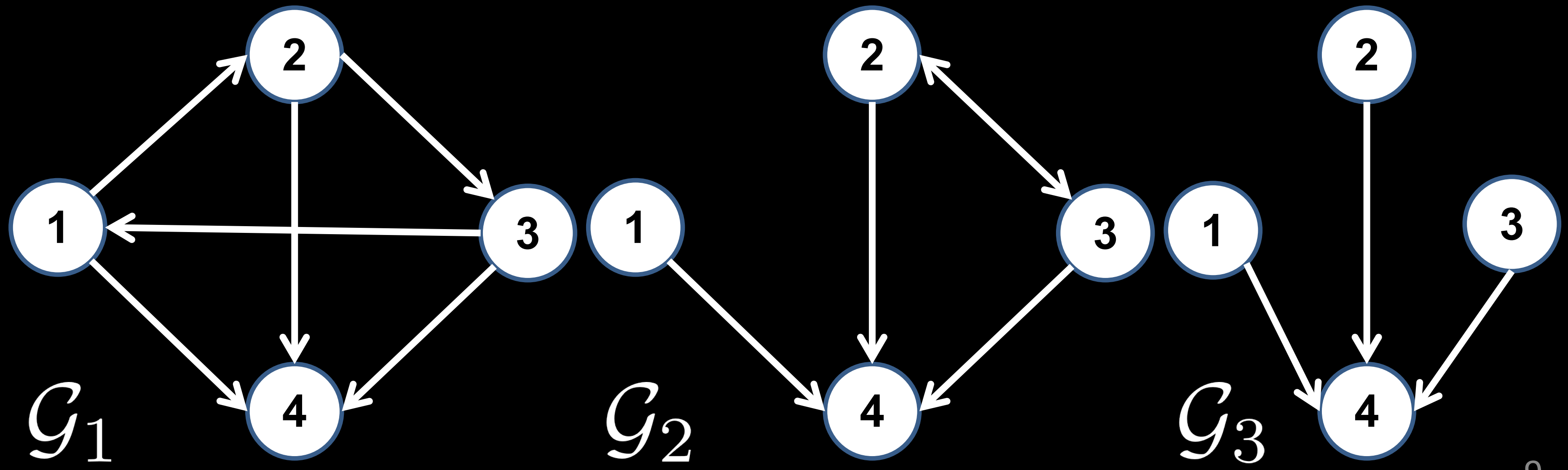
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

if $\mathcal{G}$ contains a spanning k-tree

then $\mathcal{G}$ contains $i \in [1, k]$

closed strong components

example ($k = 3$):



Graph theory: matrices

Weighted graph

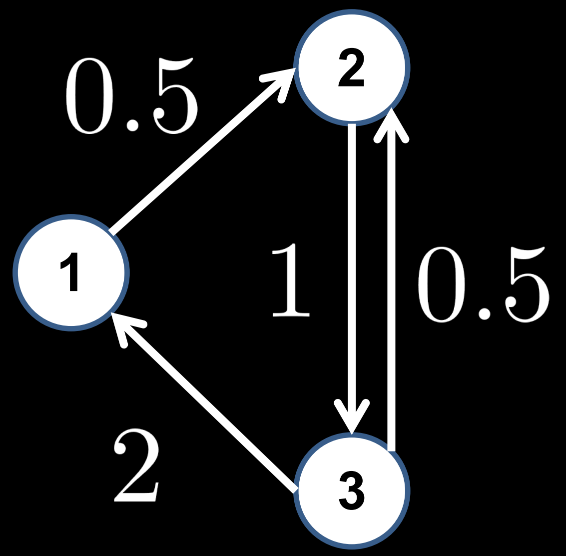
graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node set $\mathcal{V} = \{v_1, \ldots, v_n\}$

edge set $\mathcal{E} = \{(v_i, v_j), \ldots\}$

edge (v_i, v_j) has weight a_{ji}

example:



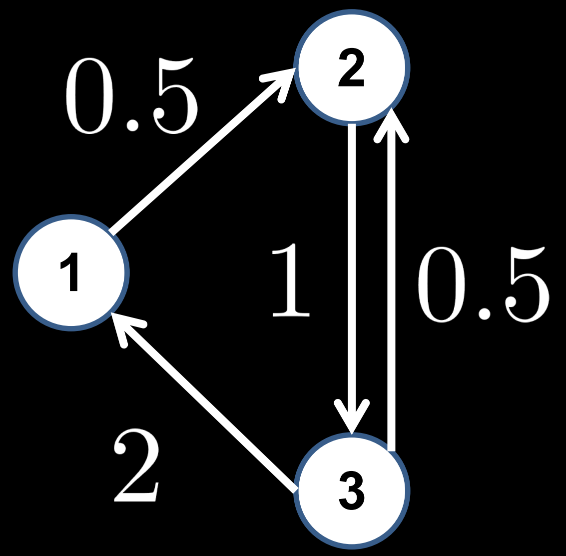
Weighted graph

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

convention:

- $(\forall i \in [1, n])(v_i, v_i) \notin \mathcal{V}$
- weight $a_{ji} = 0$ iff edge $(v_i, v_j) \notin \mathcal{V}$

example:



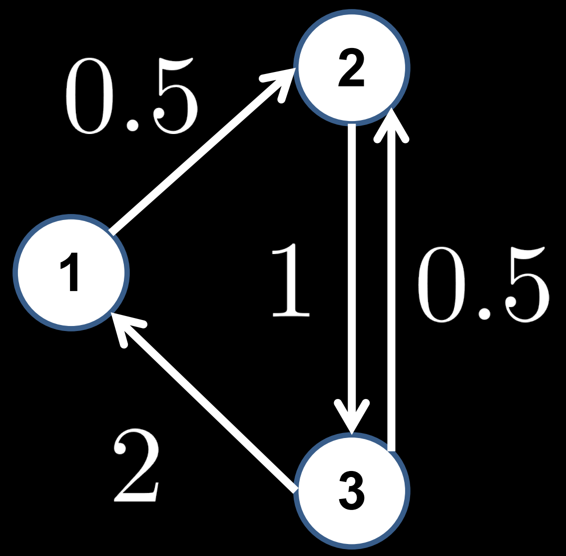
Weighted degree

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

edge (v_i, v_j) has weight a_{ji}

weighted degree of v_j is $d_{v_j} = \sum_{i \in \mathcal{N}_j} a_{ji}$

example:



Weighted degree

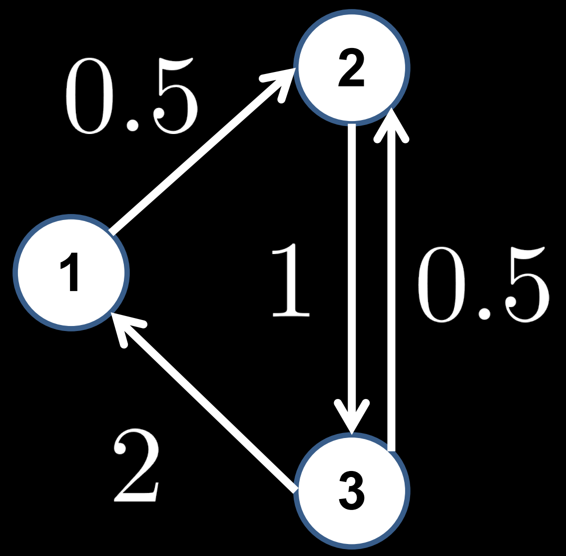
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

edge (v_i, v_j) has weight a_{ji}

weighted out-degree of v_j is

$$d^o_{v_j} = \sum_{i \in \mathcal{N}^o_j} a_{ij}$$

example:



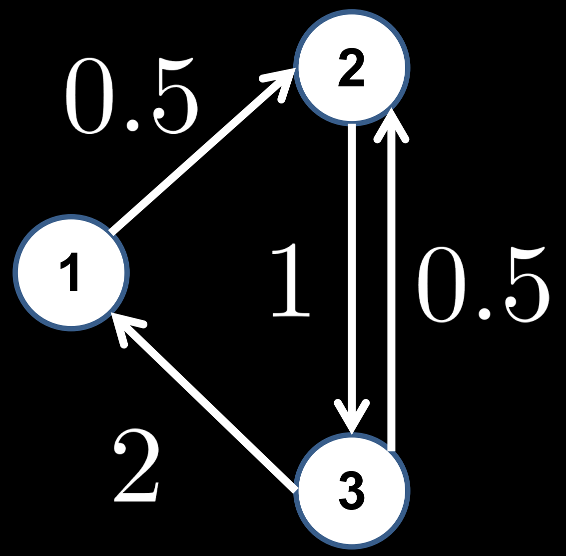
Balanced weighted graph

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

edge (v_i, v_j) has weight a_{ji}

node v_j is weight-balanced if $d_{v_j} = d^o_{v_j}$

example:



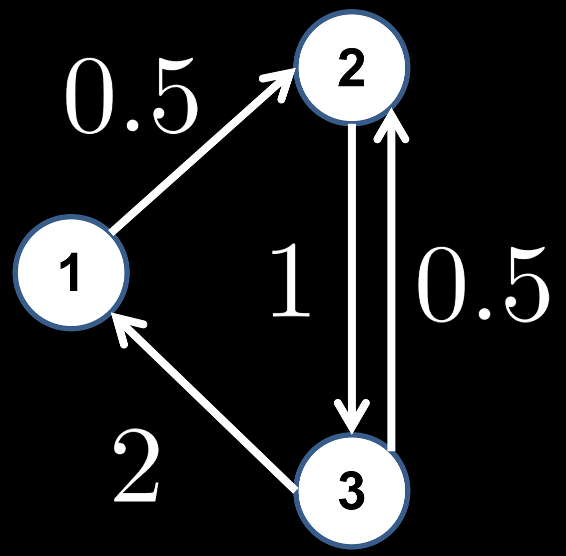
Balanced weighted graph

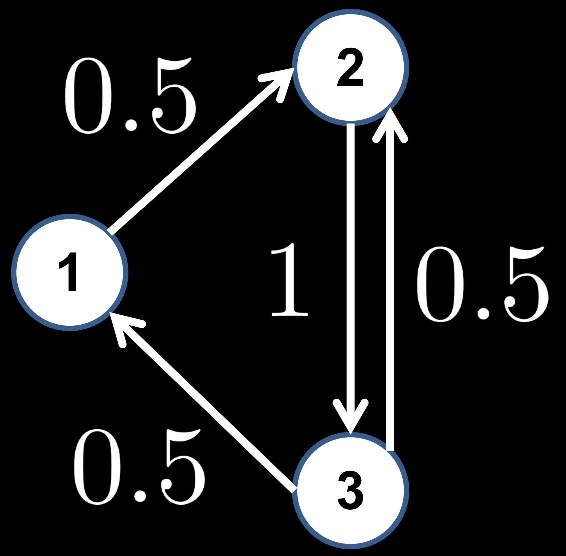
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

edge (v_i, v_j) has weight a_{ji}

$\mathcal{G}$ is weight-balanced if
every v is weight-balanced

example:





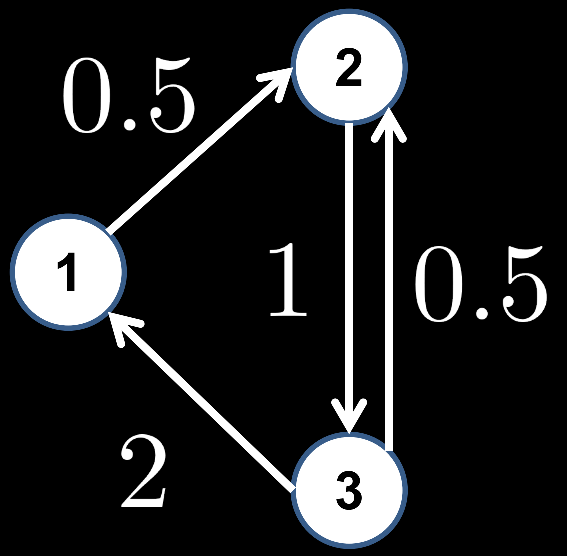
Adjacency matrix

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

edge (v_i, v_j) has weight a_{ji}

adjacency matrix $A = [a_{ij}]$

example:



Degree matrix

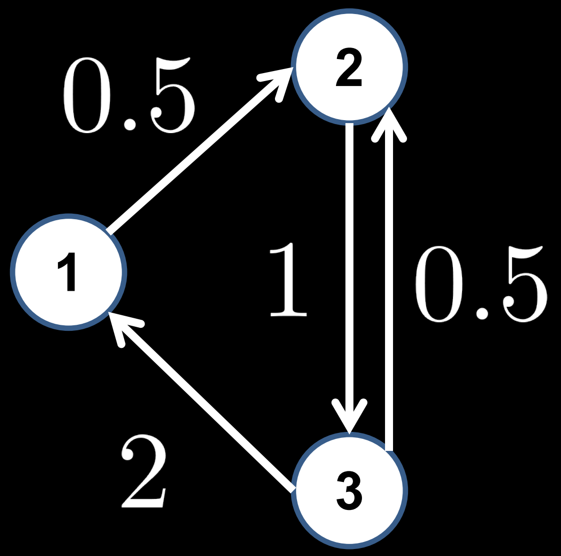
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

edge (v_i, v_j) has weight a_{ji}

degree matrix $D = \mathrm{diag}(d_{v_1}, \ldots, d_{v_n})$

('diag' means diagonalization)

example:



Adjacency & degree matrix

$$\underset{A}{\begin{bmatrix} 0 & 0 & 2 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}} \qquad \underset{D}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\mathrm{diag}(A\mathbf{1}) = D,\ \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

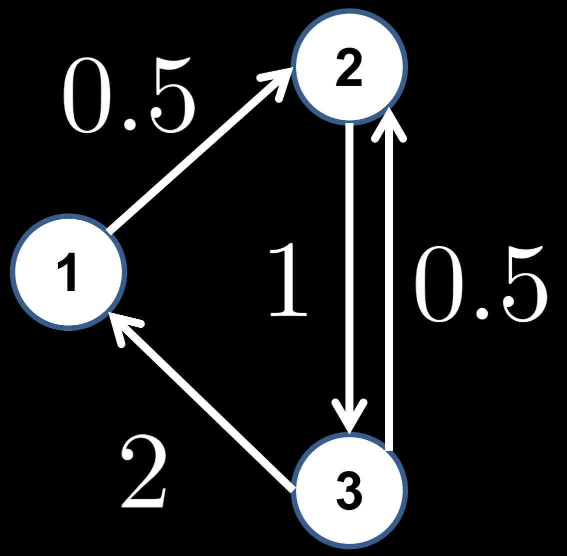
Laplacian matrix

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

edge (v_i, v_j) has weight a_{ji}

Laplacian matrix $L = D - A$

example:



Laplacian matrix

$$L = \begin{bmatrix} 2 & 0 & -2 \\ -0.5 & 1 & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Every row sums up to zero

$$L\mathbf{1} = (D - A)\mathbf{1}$$

Eigenvalue & eigenvector

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

L has an eigenvalue 0,
with eigenvector $\mathbf{1}$ (?)

Rank

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$$\operatorname{rank}(L) \leq n - 1$$

if $\mathcal{G}$ contains a spanning tree

$$\operatorname{rank}(L) \geq n - 1$$

if $\mathcal{G}$ contains a spanning 2-tree

$$\operatorname{rank}(L) \geq n - 2$$

if $\mathcal{G}$ contains a spanning k-tree

$$\operatorname{rank}(L) \geq n - k$$

Rank vs. spanning tree

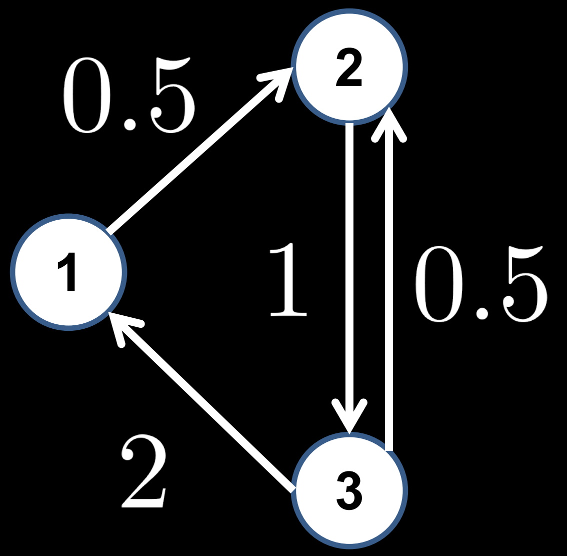
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

if $\mathcal{G}$ contains a spanning tree

$$\mathrm{rank}(L) = n - 1$$

example:



$$L = \begin{bmatrix} 2 & 0 & -2 \\ -0.5 & 1 & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Rank vs. spanning 2-tree

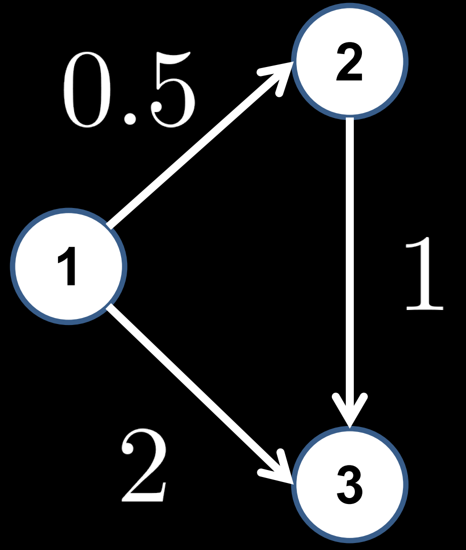
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

if $\mathcal{G}$ contains a spanning 2-tree

$$n - 2 \le \text{rank}(L) \le n - 1$$

example:



$$L = \begin{bmatrix} 0 & 0 & 0 \\ -0.5 & 0.5 & 0 \\ -2 & -1 & 3 \end{bmatrix}$$

Rank vs. spanning 2-tree

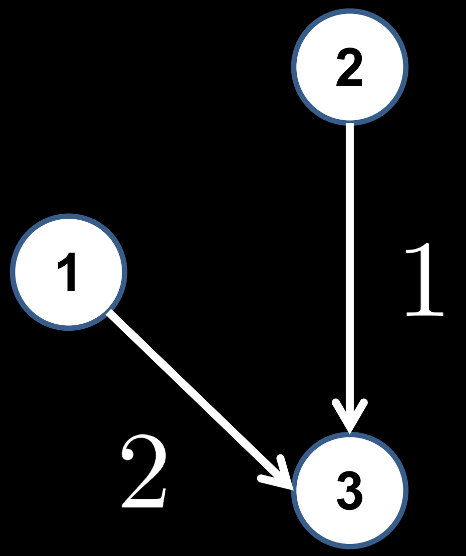
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

if $\mathcal{G}$ contains a spanning 2-tree

$$n - 2 \leq \text{rank}(L) \leq n - 1$$

example:



$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -1 & 3 \end{bmatrix}$$

Rank vs. spanning *k*-tree

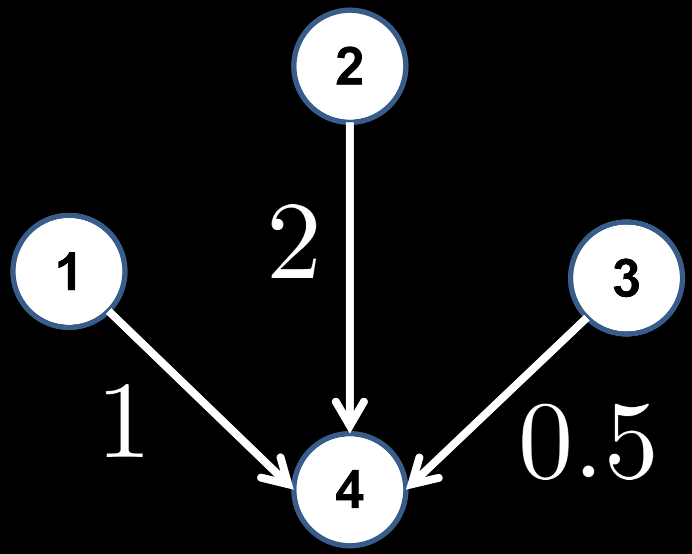
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

if $\mathcal{G}$ contains a spanning k-tree

$$n - k \leq \mathrm{rank}(L) \leq n - 1$$

example:



$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -0.5 & 3.5 \end{bmatrix}$$

Rank vs. spanning *k*-tree

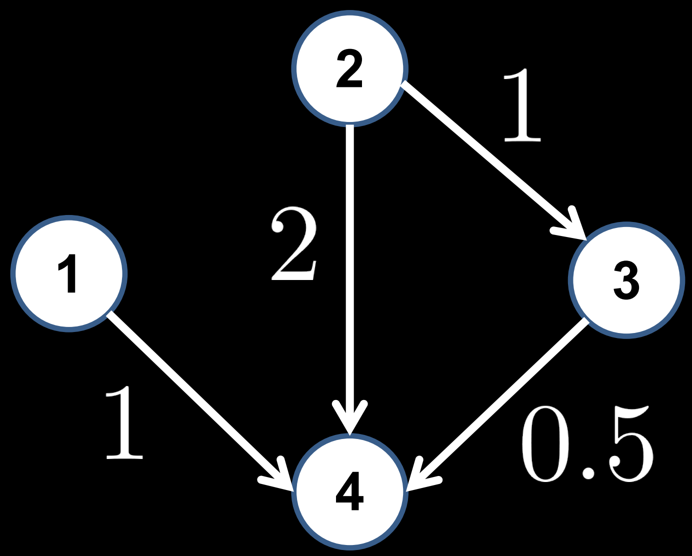
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

if $\mathcal{G}$ contains a spanning k-tree

$$n - k \leq \mathrm{rank}(L) \leq n - 1$$

example:



$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & -0.5 & 3.5 \end{bmatrix}$$

Rank vs. spanning *k*-tree

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

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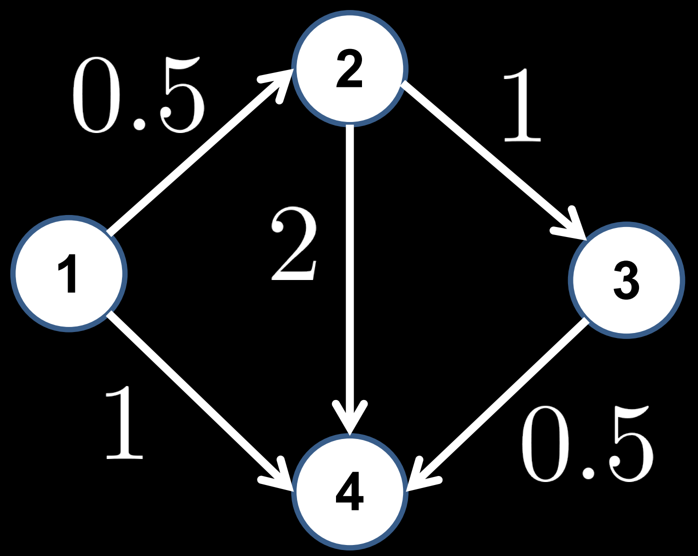
example:



$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & -0.5 & 3.5 \end{bmatrix}$$