

Multi-Agent Systems

Kai Cai

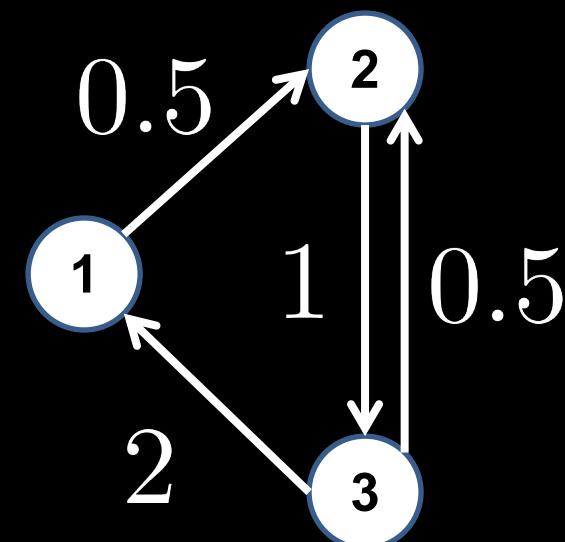
cai@omu.ac.jp

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

adjacency matrix A

example:



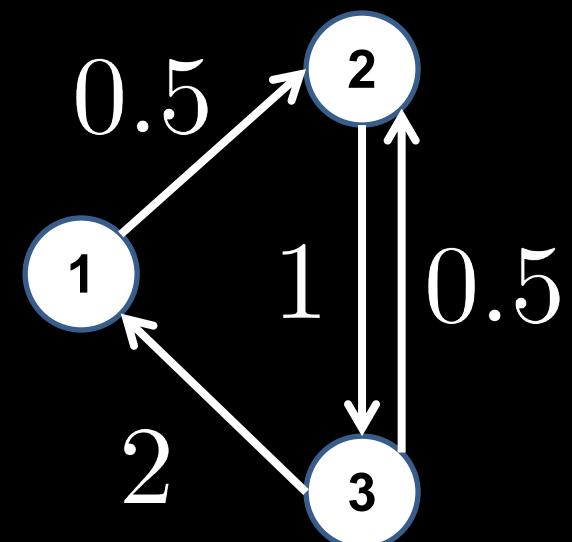
$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

degree matrix D

example:



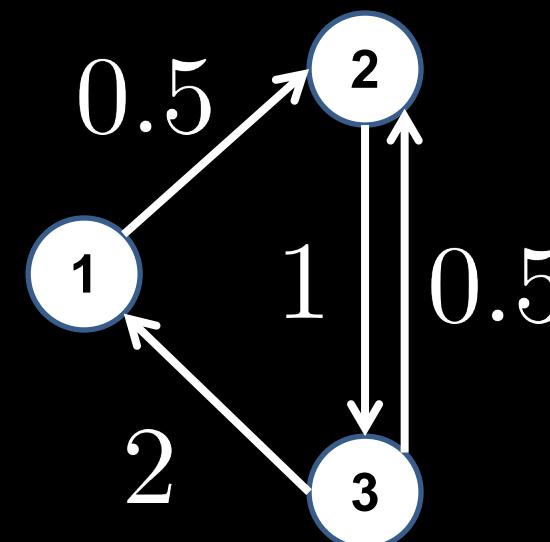
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Laplacian matrix $L = D - A$

example:



$$L = \begin{bmatrix} 2 & 0 & -2 \\ -0.5 & 1 & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Laplacian matrix $L = D - A$

Every row of L sums up to zero: $L\mathbf{1} = 0$

L has an eigenvalue 0,
with eigenvector $\mathbf{1}$

$\text{rank}(L) \leq n - 1$

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Laplacian matrix $L = D - A$

if \mathcal{G} contains a spanning tree

$$\text{rank}(L) = n - 1$$

if \mathcal{G} contains a spanning 2-tree

$$n - 2 \leq \text{rank}(L) \leq n - 1$$

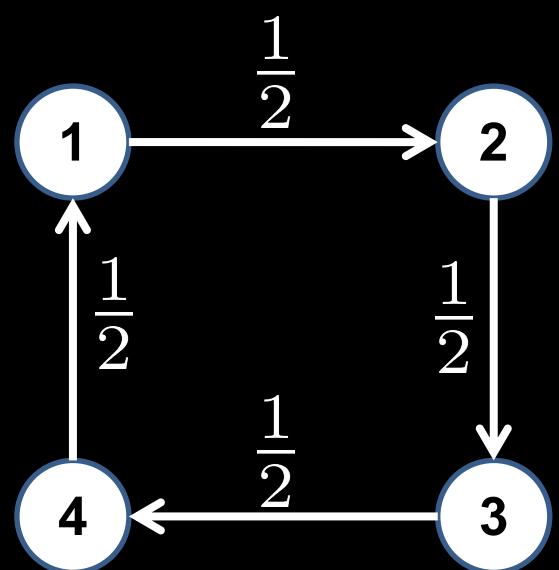
if \mathcal{G} contains a spanning k -tree

$$n - k \leq \text{rank}(L) \leq n - 1$$

Nonnegative matrix

matrix A is nonnegative, $A \geq 0$
if every entry $a_{ij} \geq 0$

example:



adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \geq 0$$

Positive matrix

matrix A is positive, $A > 0$

if every entry $a_{ij} > 0$

example:

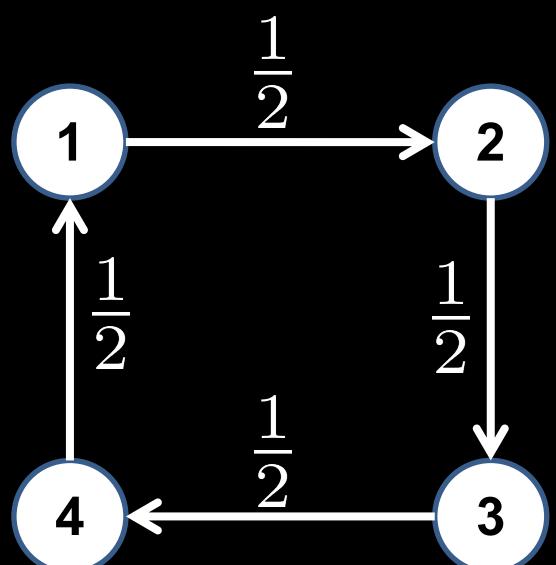
$$A = \begin{bmatrix} 1 & 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & 1 \\ 1 & \frac{1}{2} & 2 & 2 \\ 3 & \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix} > 0$$

Irreducible matrix

$A \geq 0$ is an irreducible matrix

if $I + A + \cdots + A^{n-1} > 0$
(n is size of A)

example:



$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \geq 0$$

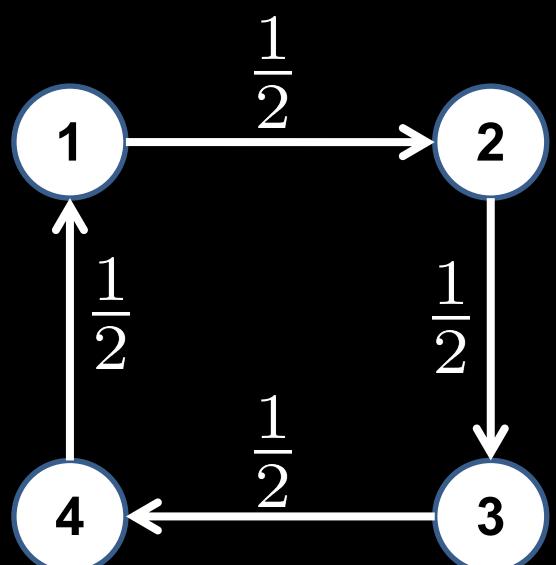
$$I + A + A^2 + A^3 = \begin{bmatrix} \frac{15}{8} & \frac{1}{8} & \frac{5}{8} & \frac{11}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{5}{8} & \frac{11}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{8} \end{bmatrix} > 0$$

Irreducible matrix

Fact: $A \geq 0$ is adjacency matrix of \mathcal{G} .

A is an irreducible matrix
iff \mathcal{G} is strongly connected

example:



$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \geq 0$$

$$I + A + A^2 + A^3 = \begin{bmatrix} \frac{15}{8} & \frac{1}{8} & \frac{5}{8} & \frac{11}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{5}{8} & \frac{11}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{11}{8} & \frac{15}{8} \end{bmatrix} > 0$$

Primitive matrix

$A \geq 0$ is a primitive matrix

if $(\exists k > 0) A^k > 0$

example:

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \geq 0$$

$$A^3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} > 0$$

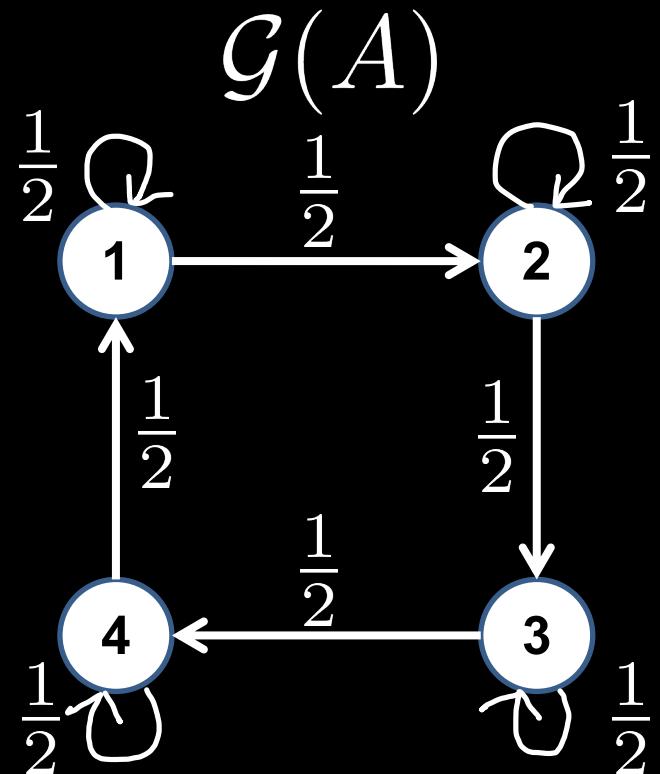
Primitive matrix

Fact: $A \geq 0$ is adjacency matrix of \mathcal{G}
(every node has a selfloop edge).

A is a primitive matrix

iff \mathcal{G} is strongly connected

example:



$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \geq 0$$

$$A^3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} > 0$$

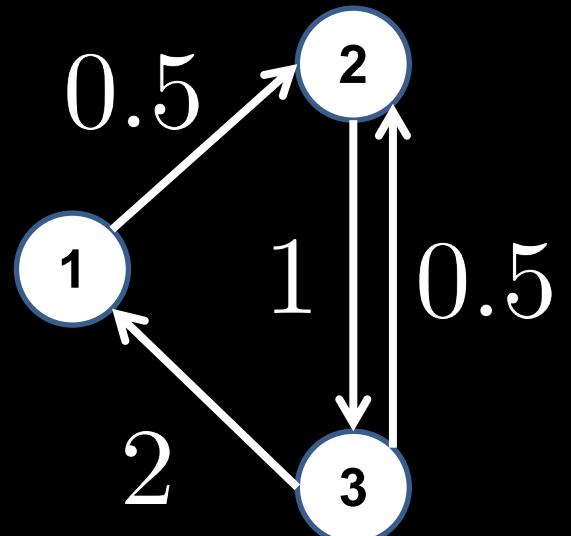
Types of Laplacian matrix

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$a_{ij} \geq 0 \Rightarrow A$ nonnegative
 $\Rightarrow L$ ordinary

example:



$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & -2 \\ -0.5 & 1 & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Types of Laplacian matrix

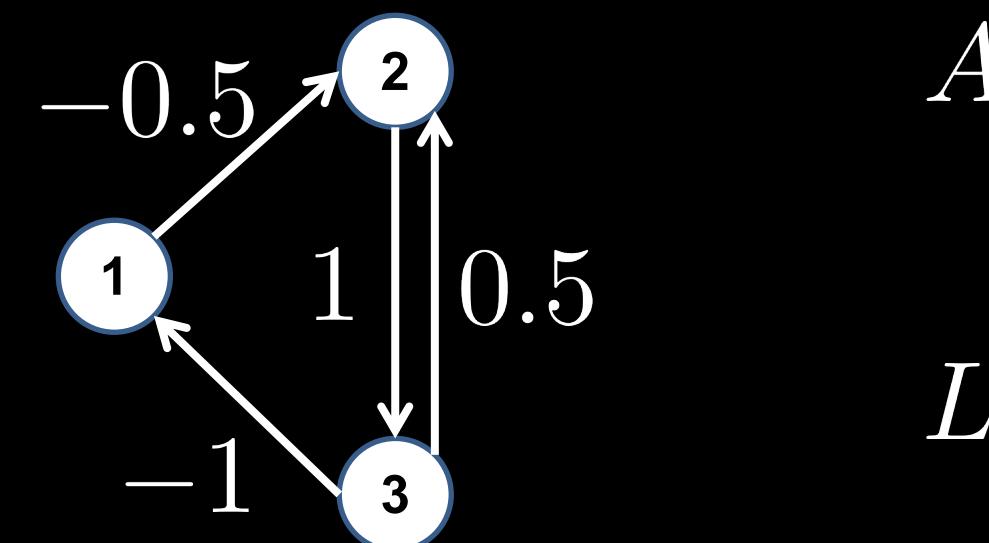
weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$a_{ij} \in \mathbb{R} \Rightarrow A$ real

$\Rightarrow L$ signed

example:



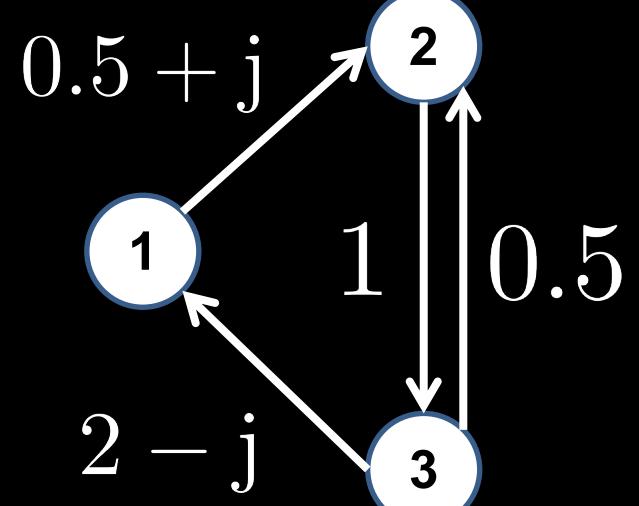
Types of Laplacian matrix

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$a_{ij} \in \mathbb{C} \Rightarrow A$ complex
 $\Rightarrow L$ complex

example:



$$A = \begin{bmatrix} 0 & 0 & 2-j \\ 0.5+j & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$
$$L = \begin{bmatrix} 2-j & 0 & j-2 \\ -0.5-j & 1+j & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Types of Laplacian matrix

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$a_{ij} \geq 0 \Rightarrow A$ nonnegative $\Rightarrow L$ ordinary
averaging, optimization, consensus

$a_{ij} \in \mathbb{C} \Rightarrow A$ complex $\Rightarrow L$ complex
2D formation control

$a_{ij} \in \mathbb{R} \Rightarrow A$ real $\Rightarrow L$ signed
3D formation control

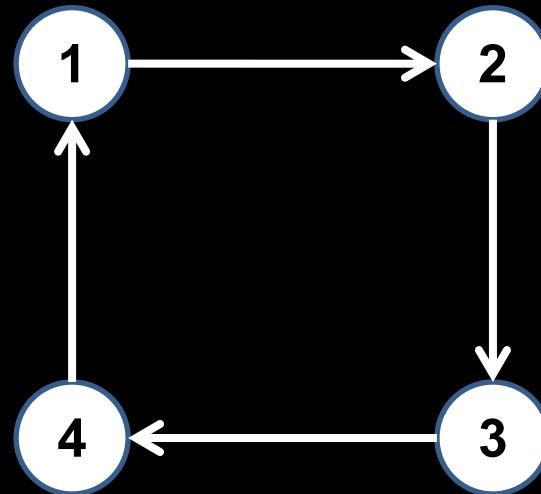
Averaging

Multi-agent system

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

node $v_i \in \mathcal{V}$: an agent
edge $(v_j, v_i) \in \mathcal{E}$: agent v_j sends
information to v_i

example:



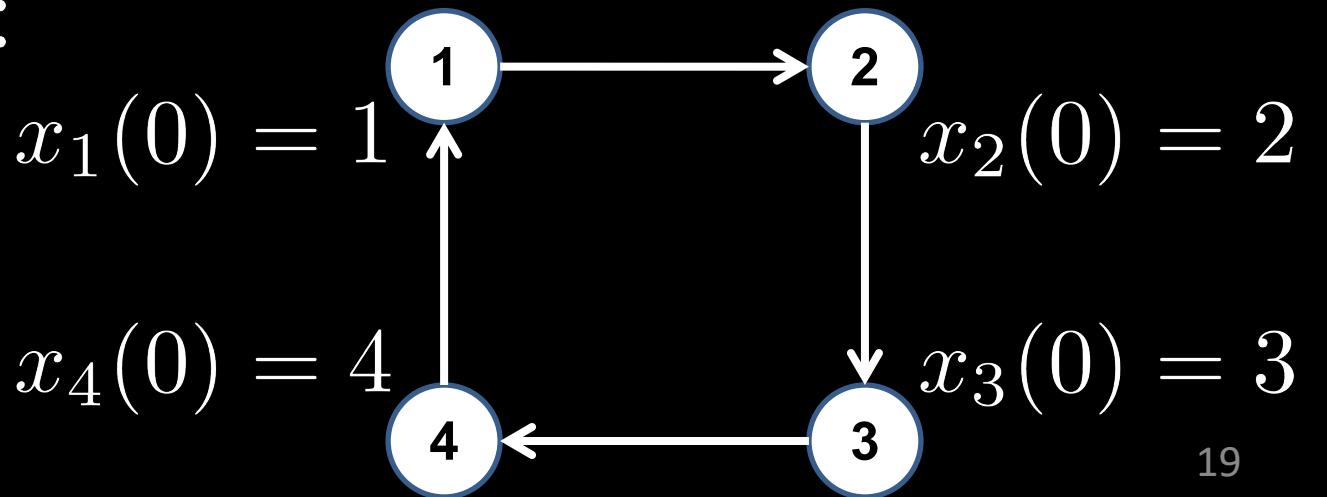
Averaging problem

each agent v_i has an initial value $x_i(0)$

averaging: update $x_i(k)$, $k = 1, 2, \dots,$

s.t. $x_i(k) \rightarrow \frac{x_1(0)+x_2(0)+x_3(0)+x_4(0)}{4} = 2.5$

example:

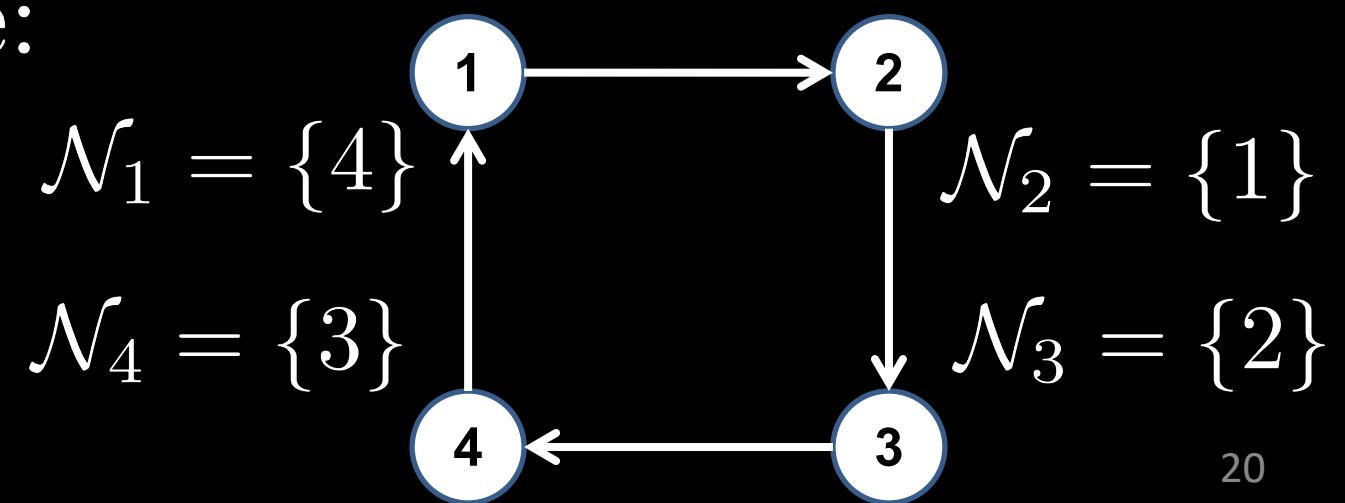


Distributed algorithm

each agent v_i can receive information $x_j(k)$ from neighbor(s) $j \in \mathcal{N}_i$

distributed algorithm: at time $k(\geq 0)$
update $x_i(k)$ based on information $x_j(k)$

example:



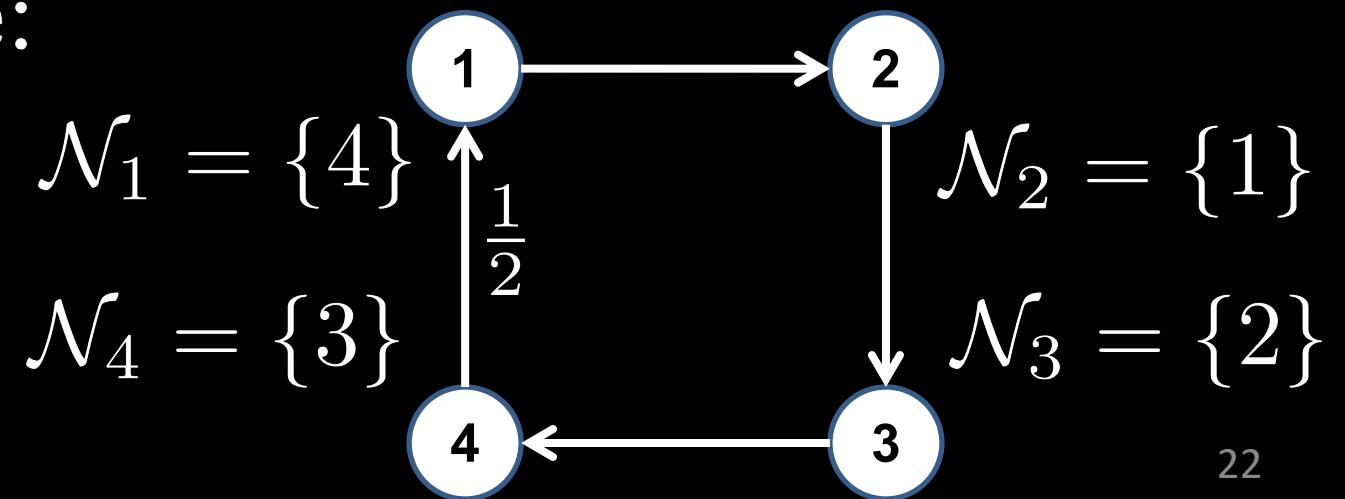
Example

$$x_1(1) = \quad x_1(0) \quad x_4(0)$$

$$x_1(2) = \frac{1}{2}(x_1(1) + x_4(1))$$

$$x_1(k+1) = \frac{1}{2}(x_1(k) + x_4(k))$$

example:



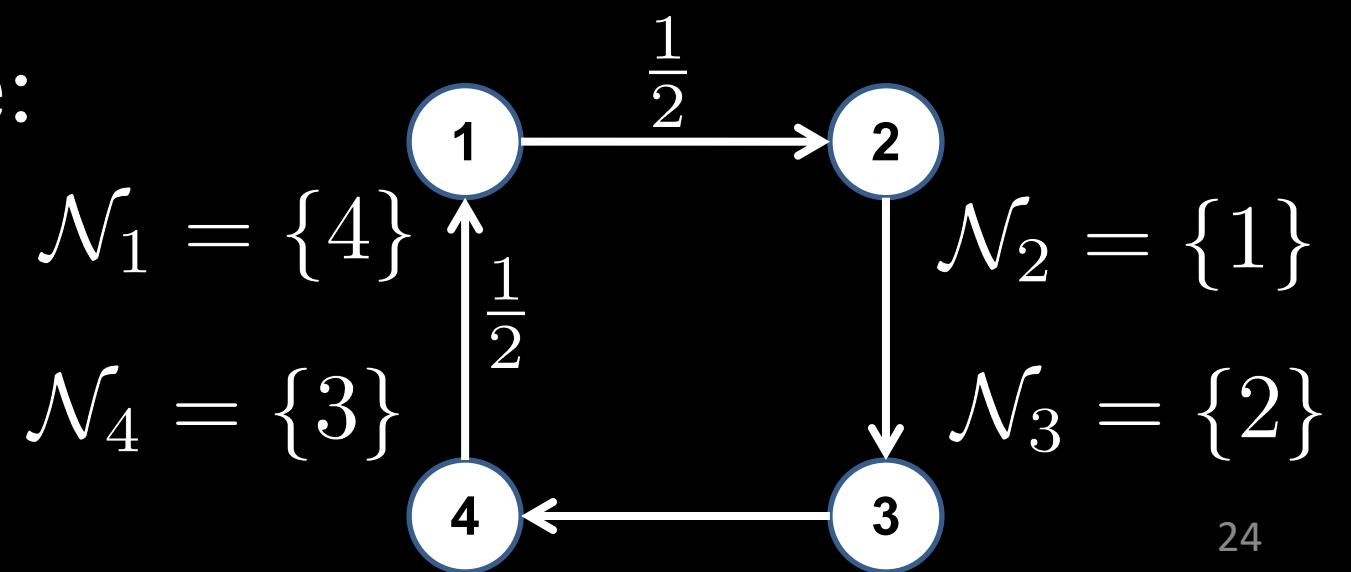
Example

$$x_2(1) = \frac{1}{2}(x_1(0) + x_2(0))$$

$$x_2(2) = \frac{1}{2}(x_1(1) + x_2(1))$$

$$x_2(k+1) = \frac{1}{2}(x_1(k) + x_2(k))$$

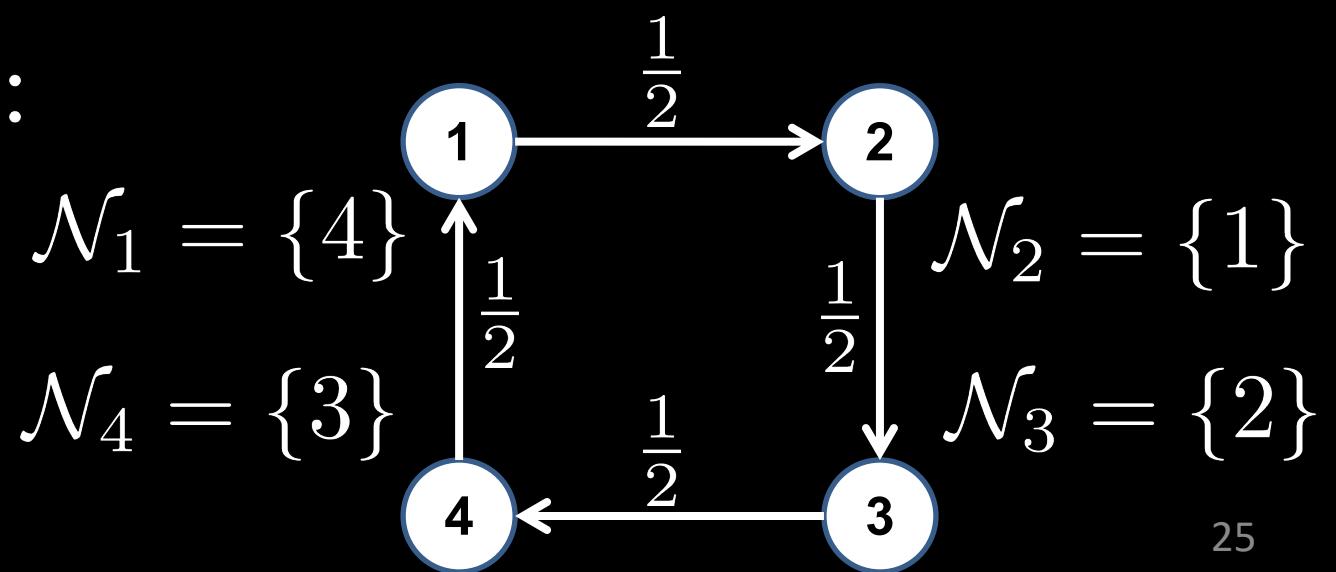
example:



Example

$$\begin{aligned}x_1(k+1) &= \frac{1}{2}(x_1(k) + x_4(k)) \\x_2(k+1) &= \frac{1}{2}(x_2(k) + x_1(k)) \\x_3(k+1) &= \frac{1}{2}(x_3(k) + x_2(k)) \\x_4(k+1) &= \frac{1}{2}(x_4(k) + x_3(k))\end{aligned}$$

example:



Example

$$x_1(k+1) = \frac{1}{2}(x_1(k) + x_4(k))$$

$$x_2(k+1) = \frac{1}{2}(x_2(k) + x_1(k))$$

$$x_3(k+1) = \frac{1}{2}(x_3(k) + x_2(k))$$

$$x_4(k+1) = \frac{1}{2}(x_4(k) + x_3(k))$$

$$x_i(k+1) = \frac{1}{1+|\mathcal{N}_i|} \left(x_i(k) + \sum_{j \in \mathcal{N}_i} x_j(k) \right)$$

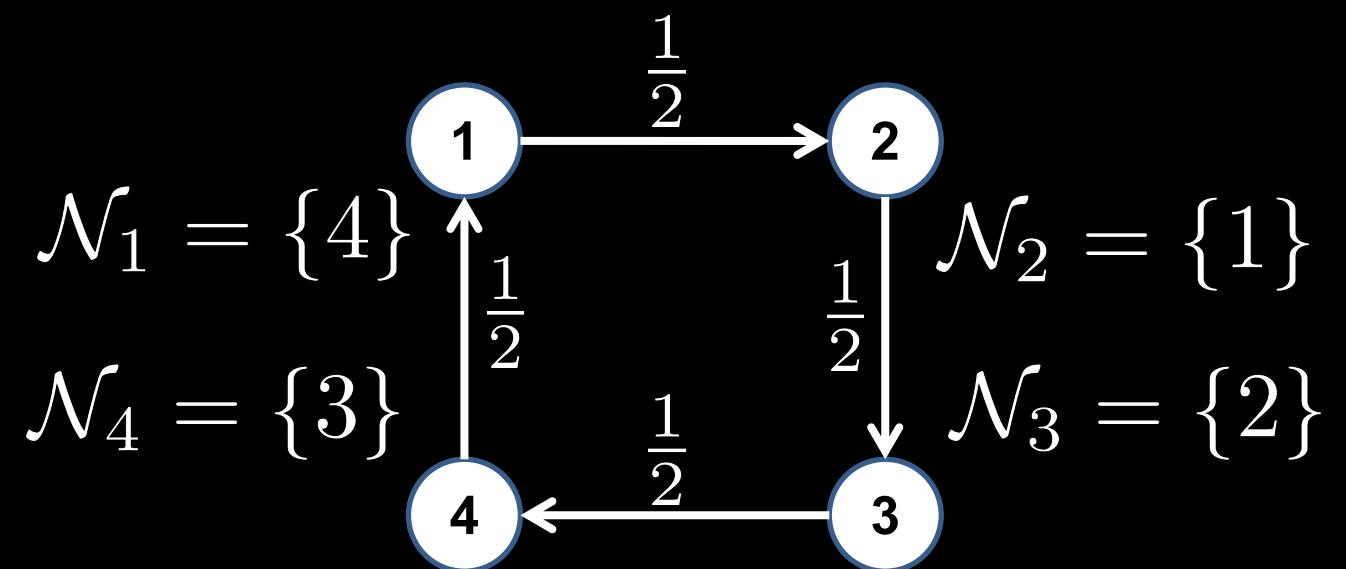
$$= \frac{1}{1+|\mathcal{N}_i|} \left((|\mathcal{N}_i| + 1)x_i(k) + \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k)) \right)$$

$$= x_i(k) + \frac{1}{1+|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k))$$

relative state information

Example

simulation: $x_1(0) = 1, x_2(0) = 2$
 $x_3(0) = 3, x_4(0) = 4$



Example

simulation: $x_1(0) = 1, x_2(0) = 2$
 $x_3(0) = 3, x_4(0) = 4$

