

Multi-Agent Systems

Kai Cai

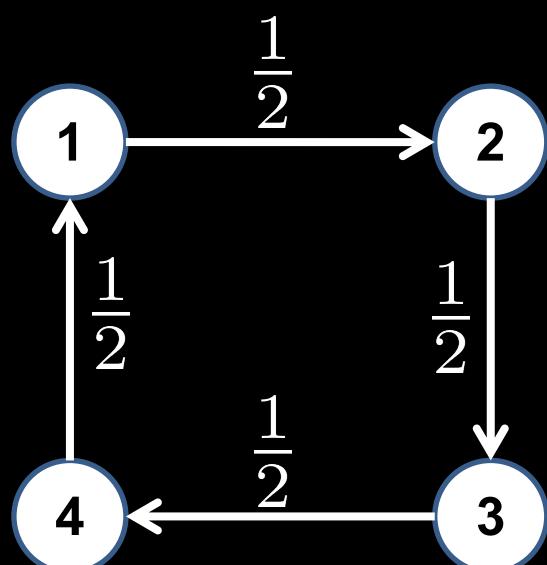
cai@omu.ac.jp

Last week

$A \geq 0$ is an irreducible matrix

if $I + A + \cdots + A^{n-1} > 0$
(n is size of A)

example:



$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \geq 0$$

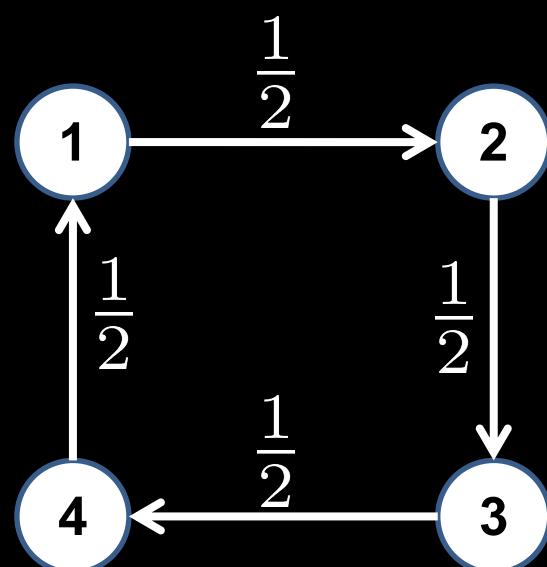
$$I + A + A^2 + A^3 = \begin{bmatrix} \frac{15}{8} & \frac{1}{8} & \frac{5}{8} & \frac{11}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{5}{8} & \frac{11}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{8} \end{bmatrix} > 0$$

Last week

Fact: $A \geq 0$ is adjacency matrix of \mathcal{G} .

A is an irreducible matrix
iff \mathcal{G} is strongly connected

example:



$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \geq 0$$

$$I + A + A^2 + A^3 = \begin{bmatrix} \frac{15}{8} & \frac{1}{8} & \frac{5}{8} & \frac{11}{8} \\ \frac{11}{8} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{5}{8} & \frac{11}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{11}{8} & \frac{15}{8} \end{bmatrix} > 0$$

Last week

$A \geq 0$ is a primitive matrix

if $(\exists k > 0) A^k > 0$

example:

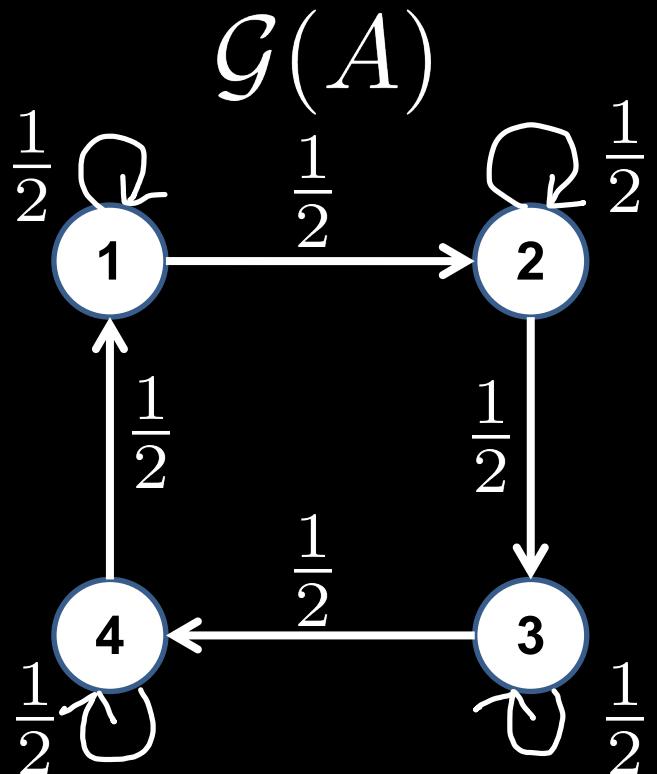
$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \geq 0$$

$$A^3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} > 0$$

Last week

Fact: $A \geq 0$ is adjacency matrix of \mathcal{G}
(every node has a selfloop edge).
 A is a primitive matrix
iff \mathcal{G} is strongly connected

example:



$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \geq 0$$

$$A^3 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} > 0$$

Last week

weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = n$

Laplacian matrix $L = D - A$

$a_{ij} \geq 0 \Rightarrow A$ nonnegative $\Rightarrow L$ ordinary
averaging, optimization, consensus

$a_{ij} \in \mathbb{C} \Rightarrow A$ complex $\Rightarrow L$ complex
2D formation control

$a_{ij} \in \mathbb{R} \Rightarrow A$ real $\Rightarrow L$ signed
3D formation control

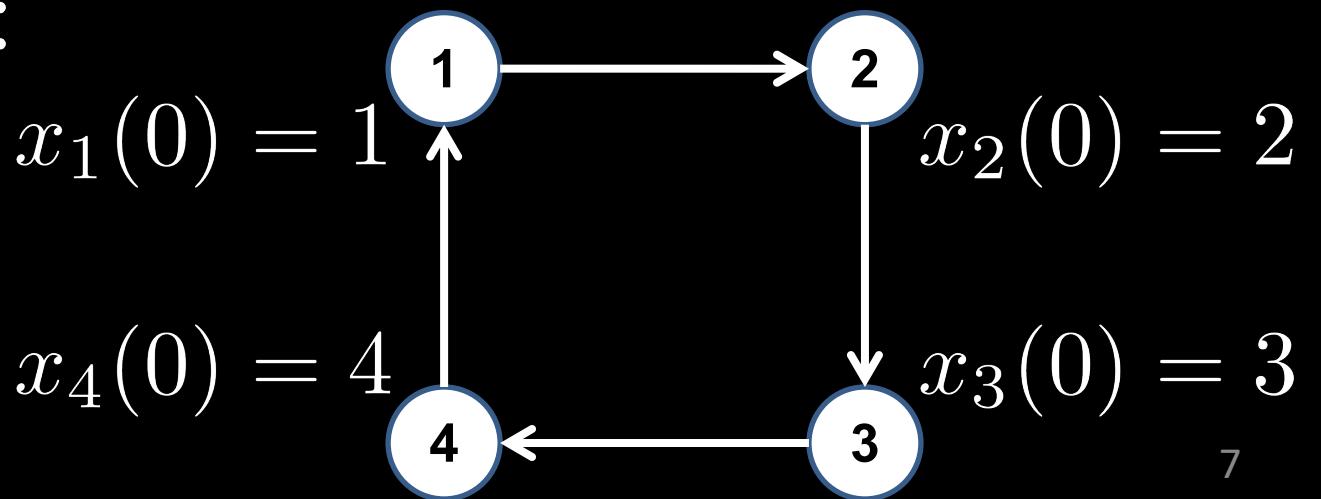
Last week

each agent v_i has an initial value $x_i(0)$

averaging: update $x_i(k)$, $k = 1, 2, \dots,$

s.t. $x_i(k) \rightarrow \frac{x_1(0)+x_2(0)+x_3(0)+x_4(0)}{4} = 2.5$

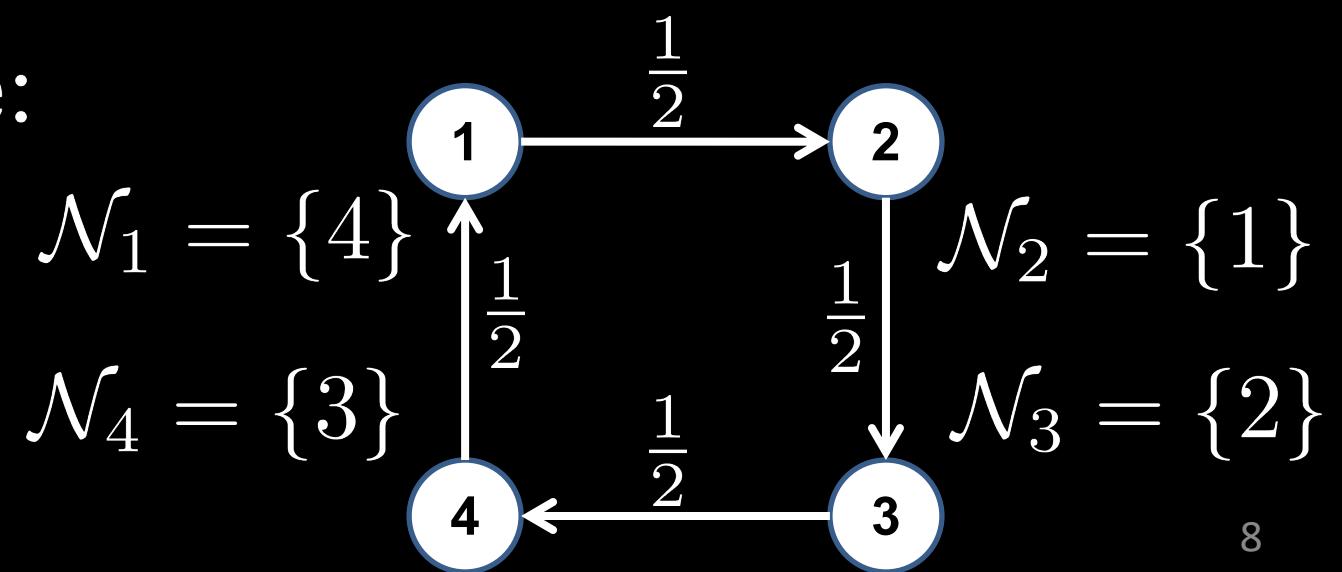
example:



Last week

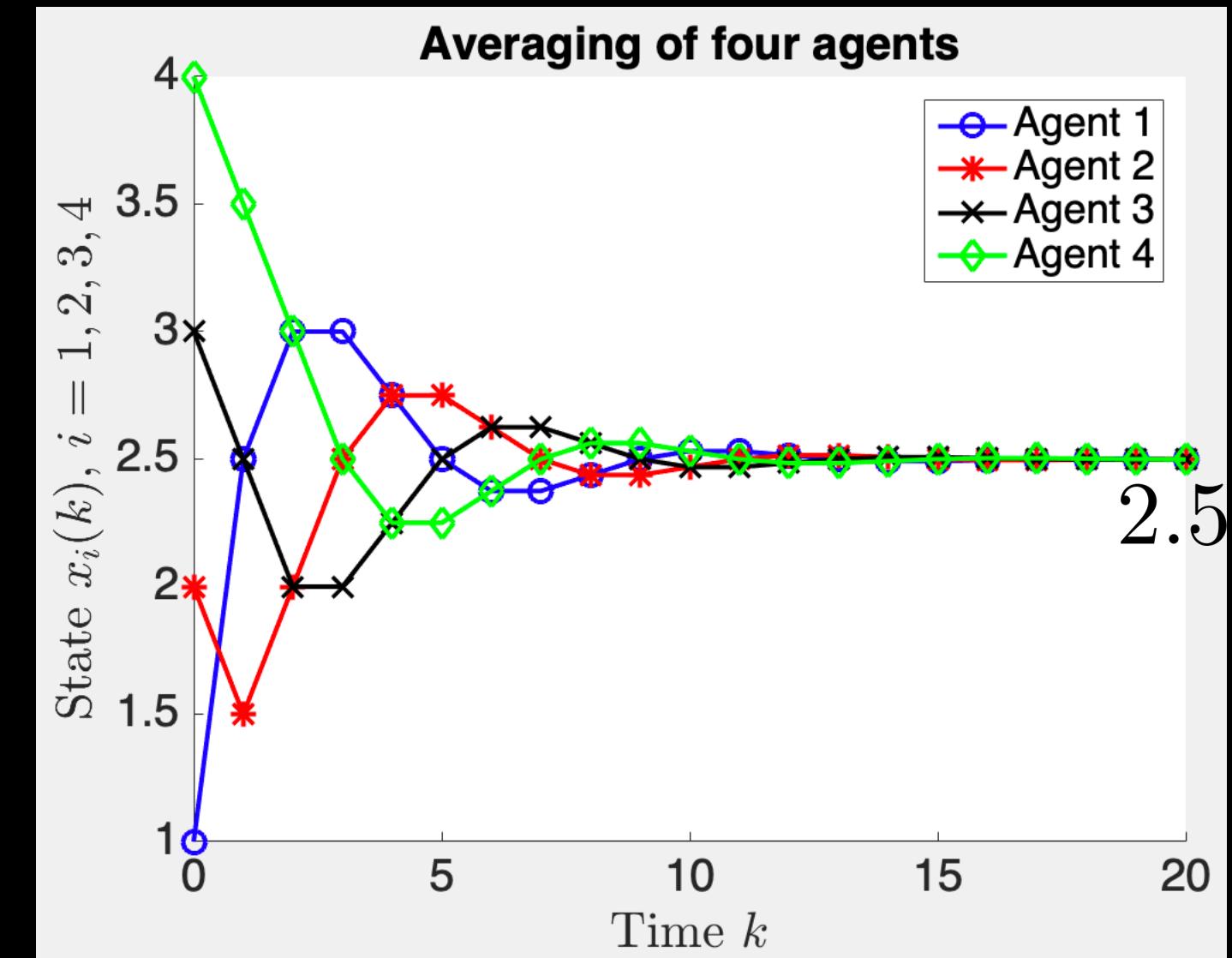
$$\begin{aligned}x_1(k+1) &= \frac{1}{2}(x_1(k) + x_4(k)) \\x_2(k+1) &= \frac{1}{2}(x_2(k) + x_1(k)) \\x_3(k+1) &= \frac{1}{2}(x_3(k) + x_2(k)) \\x_4(k+1) &= \frac{1}{2}(x_4(k) + x_3(k))\end{aligned}$$

example:



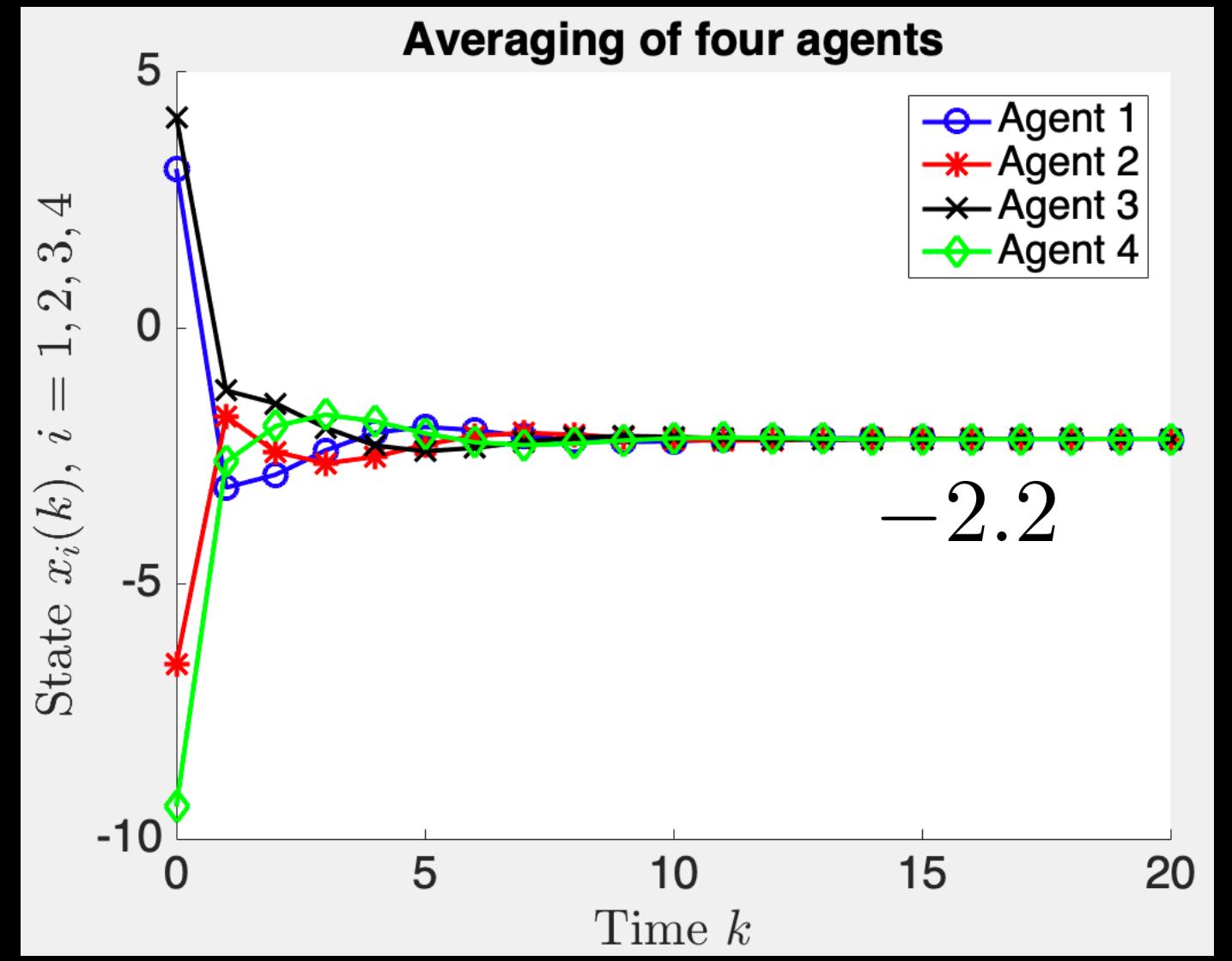
Last week

simulation: $x_1(0) = 1, x_2(0) = 2$
 $x_3(0) = 3, x_4(0) = 4$



Example

simulation: $x_1(0) = 3.1, x_2(0) = -6.6$
 $x_3(0) = 4.1, x_4(0) = -9.4$



Generalize update rule

$$x_1(k+1) = \frac{1}{2}(x_1(k) + x_4(k))$$

$$x_2(k+1) = \frac{1}{2}(x_2(k) + x_1(k))$$

$$x_3(k+1) = \frac{1}{2}(x_3(k) + x_2(k))$$

$$x_4(k+1) = \frac{1}{2}(x_4(k) + x_3(k))$$

$$x_i(k+1) = \frac{1}{1+|\mathcal{N}_i|} \left(x_i(k) + \sum_{j \in \mathcal{N}_i} \underline{x_j(k)} \right)$$

absolute state information

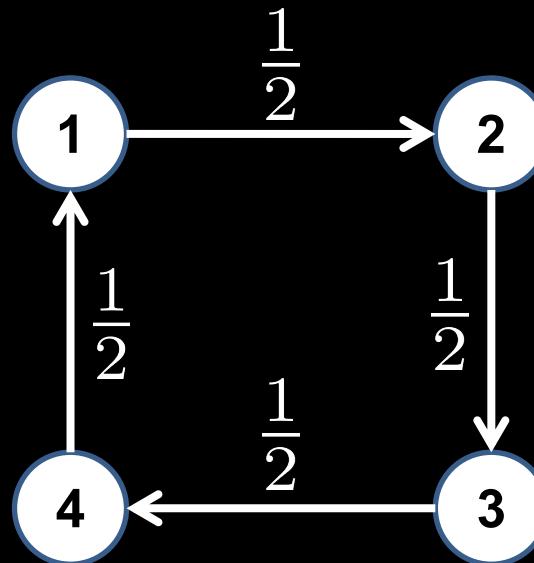
$$= x_i(k) + \frac{1}{1+|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \underline{(x_j(k) - x_i(k))}$$

relative state information

Weighted graph

example:

weighted graph \mathcal{G}



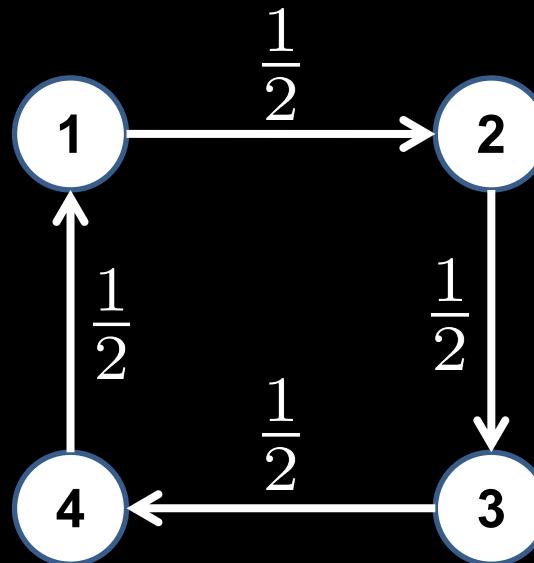
adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Weighted graph

example:

weighted graph \mathcal{G}



Laplacian matrix

$$L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Equation

$$x_1(k+1) = \frac{1}{2}(x_1(k) + x_4(k))$$

$$x_2(k+1) = \frac{1}{2}(x_2(k) + x_1(k))$$

$$x_3(k+1) = \frac{1}{2}(x_3(k) + x_2(k))$$

$$x_4(k+1) = \frac{1}{2}(x_4(k) + x_3(k))$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

Equation

$$L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$$I - L$$

Recap, generalization

- a system of n interacting agents is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- each agent v_i has an initial value $x_i(0)$

Problem: update $x_i(k)$, $k = 1, 2, \dots$,

s.t. $(\forall v_i \in \mathcal{V}) \lim_{k \rightarrow \infty} x_i(k) = \frac{1}{n} \sum_{v_i \in \mathcal{V}} x_i(0)$

Recap, generalization

- a system of n interacting agents is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- each agent v_i has an initial value $x_i(0)$

Distributed algorithm

$$x_i(k+1) = x_i(k) + \frac{1}{1+|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k))$$

based on information $x_j(k)$ or
relative information $x_j(k) - x_i(k)$
from neighbor agent(s) $j \in \mathcal{N}_i$

Recap, generalization

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
each agent v_i has an initial value $x_i(0)$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x(k+1) = (I - L)x(k)$$

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

$x(k + 1) = (I - L)x(k)$ solves averaging

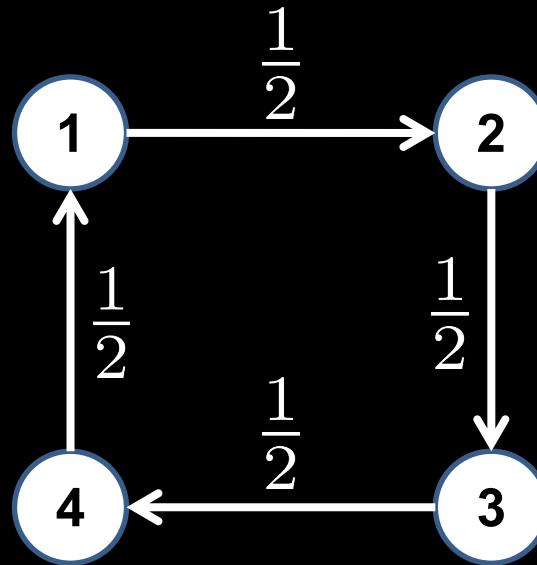
$$\text{i.e. } (\forall v_i \in \mathcal{V}) \lim_{k \rightarrow \infty} x_i(k) = \frac{1}{n} \sum_{v_i \in \mathcal{V}} x_i(0)$$

iff \mathcal{G} is strongly connected and
weight balanced

Example

example:

weighted graph \mathcal{G}



strongly connected (?)

weight balanced (?)

Example

ordinary Laplacian matrix

$$L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

strongly connected \Rightarrow spanning tree
 $\text{rank}(L) = ?$

weight balanced \Rightarrow column sum 0
 $\mathbf{1}^\top L = ?$

Example

eigenvalues

$$\begin{aligned} L &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{2} I - \frac{1}{2} M = \frac{1}{2}(I - M) \end{aligned}$$

Example

eigenvalues

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example

spectrum mapping

$$L = \frac{1}{2}(I - M)$$

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

spectrum mapping

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

nonnegative entries $\Rightarrow I - L \geq 0$

row sum is 1 $\Rightarrow (I - L)\mathbf{1} = \mathbf{1}$

“row-stochastic matrix”

1 is an eig of $I - L$, with eigvec 1

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

nonnegative entries $\Rightarrow I - L \geq 0$

column sum is 1 $\Rightarrow \mathbf{1}^\top (I - L) = \mathbf{1}^\top$

“column-stochastic matrix”

1 is an eig of $I - L$, with left eigvec 1

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

nonnegative entries $\Rightarrow I - L \geq 0$

row sum is 1 $\Rightarrow (I - L)\mathbf{1} = \mathbf{1}$

column sum is 1 $\Rightarrow \mathbf{1}^\top (I - L) = \mathbf{1}^\top$

“doubly-stochastic matrix”

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

row (or column) stochastic matrix

spectral radius $\rho(I - L) = 1$

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is a primitive matrix because

$$\exists k = 3 \text{ s.t. } (I - L)^k = (I - L)^3 =$$

$$\begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & 1 & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & 1 & \frac{1}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{3}{8} & 1 \end{bmatrix} > 0$$

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is a primitive matrix because
graph $\mathcal{G}(I - L)$ is strongly connected
(every node of $\mathcal{G}(I - L)$ is selflooped)

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Perron-Frobenius Theorem

If $I - L$ is nonnegative and primitive,
 $\rho(I - L)$ is a simple eigenvalue of $I - L$
 $\rho(I - L) > |\lambda|$ for all other eigenvalues λ

Example

eigenvalues

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Perron-Frobenius Theorem

$$\lambda_1 = \rho(I - L) = 1$$

$$\lambda_2 = \frac{1}{2} + \frac{1}{2}j \quad (|\lambda_2| < 1)$$

$$\lambda_3 = \frac{1}{2} - \frac{1}{2}j \quad (|\lambda_3| < 1)$$

$$\lambda_4 = 0 \quad (|\lambda_4| < 1)$$