Multi-Agent Systems

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Last week

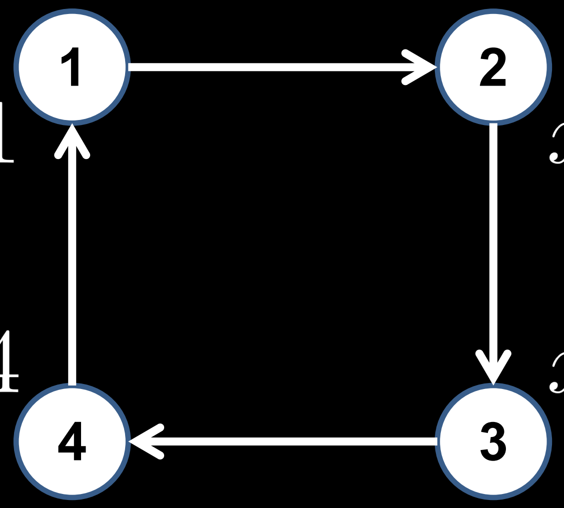
each agent v_i has an initial value $x_i(0)$

averaging: update $x_i(k)$, $k = 1, 2, \ldots,$

s.t. $x_i(k) \to \frac{x_1(0)+x_2(0)+x_3(0)+x_4(0)}{4} = 2.5$

example:

$x_1(0) = 1$ $x_2(0) = 2$



$x_4(0) = 4$ $x_3(0) = 3$

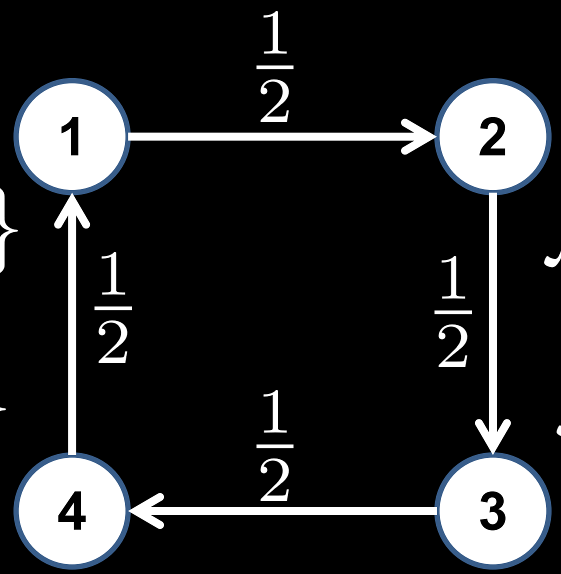
Last week

$$
\begin{aligned}
x_1(k+1) &= \tfrac{1}{2}(x_1(k) + x_4(k)) \\
x_2(k+1) &= \tfrac{1}{2}(x_2(k) + x_1(k)) \\
x_3(k+1) &= \tfrac{1}{2}(x_3(k) + x_2(k)) \\
x_4(k+1) &= \tfrac{1}{2}(x_4(k) + x_3(k))
\end{aligned}
$$

example:

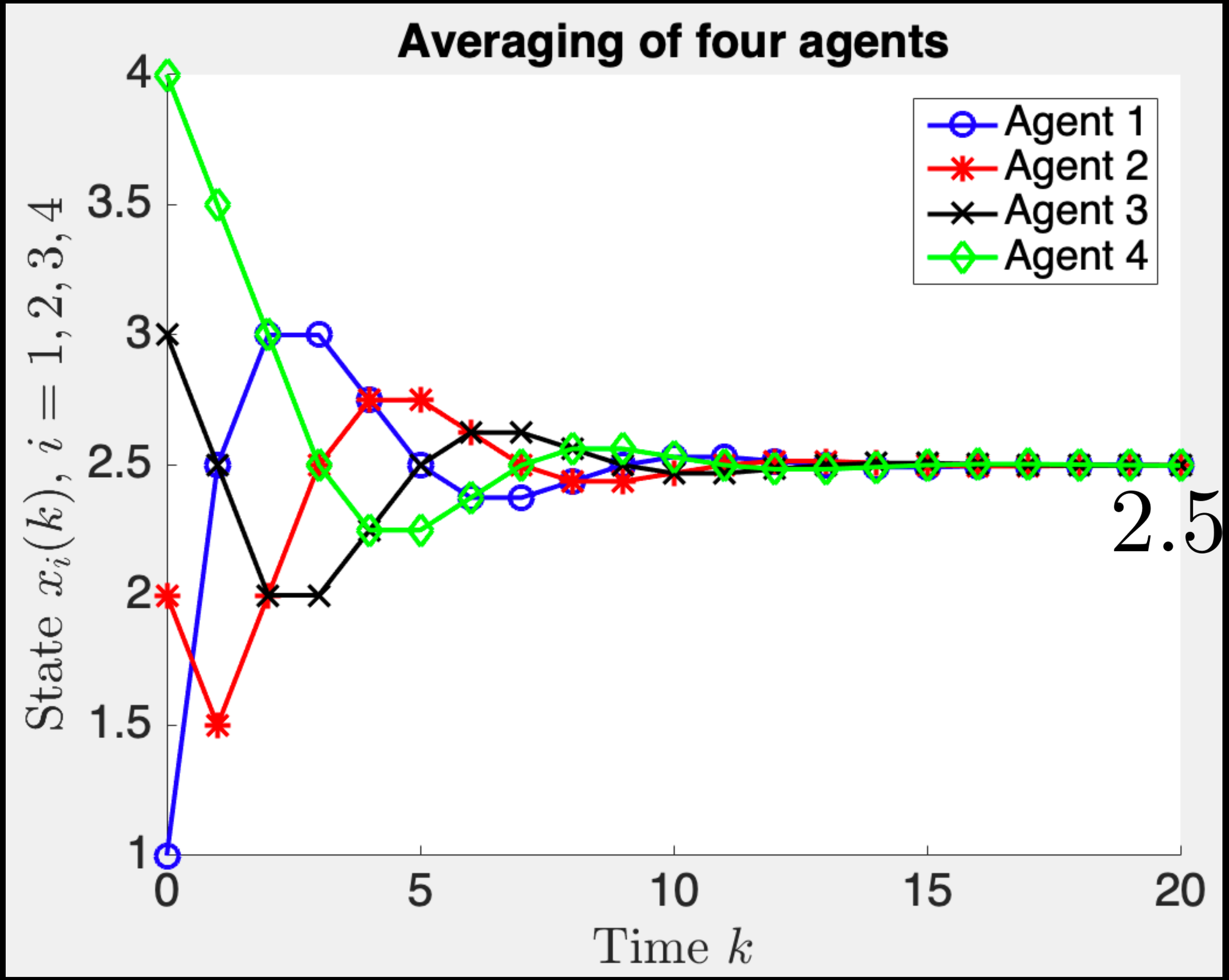
$\mathcal{N}_1 = \{4\}$ $\mathcal{N}_2 = \{1\}$

$\mathcal{N}_4 = \{3\}$ $\mathcal{N}_3 = \{2\}$



Last week

$$\text{simulation: } x_1(0) = 1, x_2(0) = 2$$
$$x_3(0) = 3, x_4(0) = 4$$



Last week

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

Problem: update $x_i(k)$, $k = 1, 2, \ldots,$

$$\text{s.t. } (\forall v_i \in \mathcal{V}) \lim_{k \to \infty} x_i(k) = \frac{1}{n} \sum_{v_i \in \mathcal{V}} x_i(0)$$

Last week

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

Distributed algorithm

$$x_{i+1}(k) = \frac{1}{1+|\mathcal{N}_i|}\left(x_i(k) + \sum_{j\in\mathcal{N}_i} x_j(k)\right)$$

based on information $x_j(k)$ or

relative information $x_j(k) - x_i(k)$

from neighbor agent(s) $j \in \mathcal{N}_i$

Last week

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{x(k+1) = (I - L)x(k)}$$

Last week: Theorem

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

$x(k+1) = (I - L)x(k)$ solves averaging

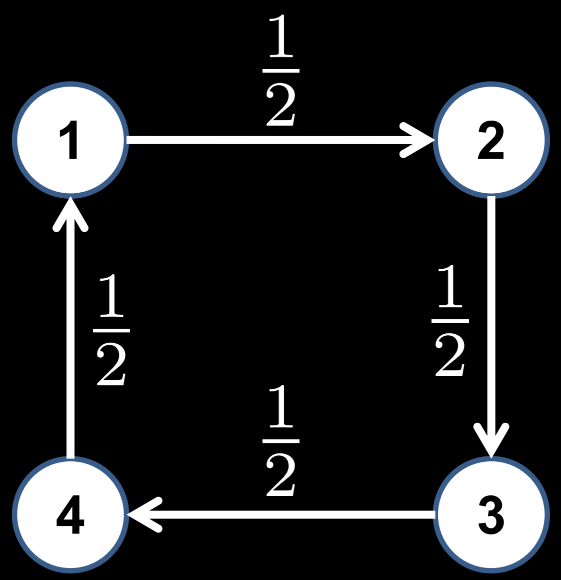
$$\text{i.e. } (\forall v_i \in \mathcal{V}) \lim_{k \to \infty} x_i(k) = \frac{1}{n} \sum_{v_i \in \mathcal{V}} x_i(0)$$

iff $\mathcal{G}$ is strongly connected and weight balanced

Last week

example:

weighted graph $\mathcal{G}$



$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

row-stochastic

column-stochastic

$$\lambda_1 = \rho(I - L) = 1$$

$$\lambda_2 = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{j} \ (|\lambda_2| < 1)$$

$$\lambda_3 = \tfrac{1}{2} - \tfrac{1}{2}\mathrm{j} \ (|\lambda_3| < 1)$$

$$\lambda_4 = 0 \ (|\lambda_4| < 1)$$

Convergence analysis

diagonalization

$$I - L = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= VJV^{-1}$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{1}{2}\mathrm{j} & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{1}{2}\mathrm{j} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \end{bmatrix}$$

$$(w_1^\top v_1 = 1,\ w_1^\top v_2 = 0,\ w_1^\top v_3 = 0,\ w_1^\top v_4 = 0)$$

$$v_1 = \mathbf{1},\ w_1 = \tfrac{1}{4}\mathbf{1}$$

Convergence analysis

$$\begin{aligned}
x(k+1) &= (I-L)x(k) \\
&= (I-L)^2 x(k-1) \\
&\vdots \\
&= (I-L)^{k+1} x(0) \\
&= (VJV^{-1})^{k+1} x(0)
\end{aligned}$$

Convergence analysis

$$x(k+1) = VJ^{k+1}V^{-1}x(0)$$

$$= [v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 1^{k+1} & 0 & 0 & 0 \\ 0 & (\frac{1}{2}+\frac{1}{2}\mathrm{j})^{k+1} & 0 & 0 \\ 0 & 0 & (\frac{1}{2}-\frac{1}{2}\mathrm{j})^{k+1} & 0 \\ 0 & 0 & 0 & 0^{k+1} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \end{bmatrix} x(0)$$

$$\to [v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ w_3^\top \\ w_4^\top \end{bmatrix} x(0), \text{ as } k \to \infty$$

$$= v_1 w_1^\top x(0) \qquad (v_1 = \mathbf{1},\ w_1 = \tfrac{1}{4}\mathbf{1})$$

$$= \frac{1}{4}\mathbf{1}^\top x(0)\mathbf{1}$$

Theorem

a system of n interacting agents
is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
each agent v_i has an initial value $x_i(0)$

$x(k+1) = (I - L)x(k)$ solves averaging
iff $\mathcal{G}$ is strongly connected and
weight balanced

Proof: (only if; necessity)
leave it as an exercise

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I - L)x(k)$ solves averaging

(i) $I - L$ is a nonnegative matrix

hint:

$$x_i(k+1) = \frac{1}{1+|\mathcal{N}_i|}\left(x_i(k) + \sum_{j\in\mathcal{N}_i} x_j(k)\right)$$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I - L)x(k)$ solves averaging

(ii) $I - L$ is row stochastic

alternative hint: $L\mathbf{1} = 0$

so spectral radius $\rho(I - L) = 1$

and 1 is an eigenvalue of $I - L$

with eigenvector $\mathbf{1}$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I - L)x(k)$ solves averaging

(iii) $I - L$ is column stochastic

hint: $\mathcal{G}$ is weight balanced

so eigenvalue 1 has a left eigenvector $\mathbf{1}$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I - L)x(k)$ solves averaging

(iv) $I - L$ is primitive

hint: $\mathcal{G}(I - L)$ is strongly connected
and every node of $\mathcal{G}(I - L)$ is selflooped

by Perron-Frobenius

$\rho(I - L) = 1$ is a simple eigenvalue of $I - L$
and all other $n - 1$ eigenvalues $\lambda_2, \ldots, \lambda_n$
satisfy $|\lambda_2| < 1, \ldots, |\lambda_n| < 1$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I - L)x(k)$ solves averaging

$$\text{(v) } x(k+1) = (I-L)^{k+1}x(0)$$
$$= (VJV^{-1})^{k+1}x(0)$$
$$= [v_1 \ v_2 \cdots v_n] \begin{bmatrix} 1^{k+1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{k+1} \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ \vdots \\ w_n^\top \end{bmatrix} x(0)$$

$$(w_1^\top v_1 = 1\text{: } v_1 = \mathbf{1},\ w_1 = \tfrac{1}{n}\mathbf{1})$$

Theorem

Proof: (if; sufficiency)

if $\mathcal{G}$ is strongly connected, weight balanced

show $x(k+1) = (I-L)x(k)$ solves averaging

$$\text{(v) } x(k+1) = (I-L)^{k+1}x(0)$$
$$= (VJV^{-1})^{k+1}x(0)$$

$$\to [v_1 \; v_2 \cdots v_n] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1^\top \\ w_2^\top \\ \vdots \\ w_n^\top \end{bmatrix} x(0), \text{ as } k \to \infty$$

$$(w_1^\top v_1 = 1\text{: } v_1 = \mathbf{1},\ w_1 = \tfrac{1}{n}\mathbf{1})$$

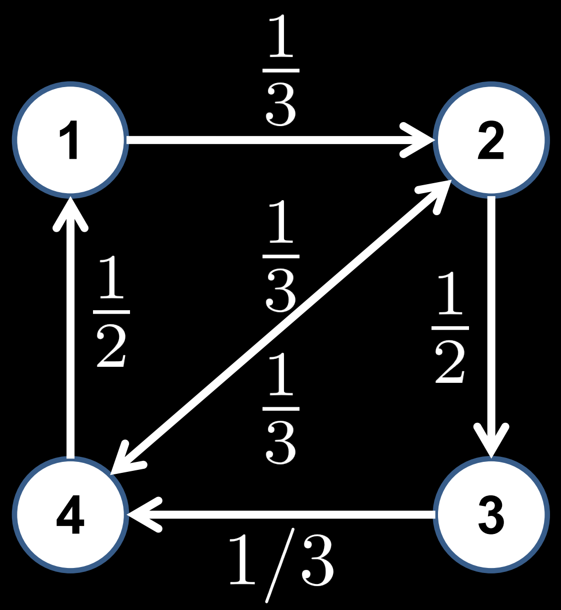
$$= v_1 w_1^\top x(0) = \frac{1}{n}\mathbf{1}^\top x(0)\mathbf{1}$$

Example

example:

weighted graph $\mathcal{G}$

not weight balanced

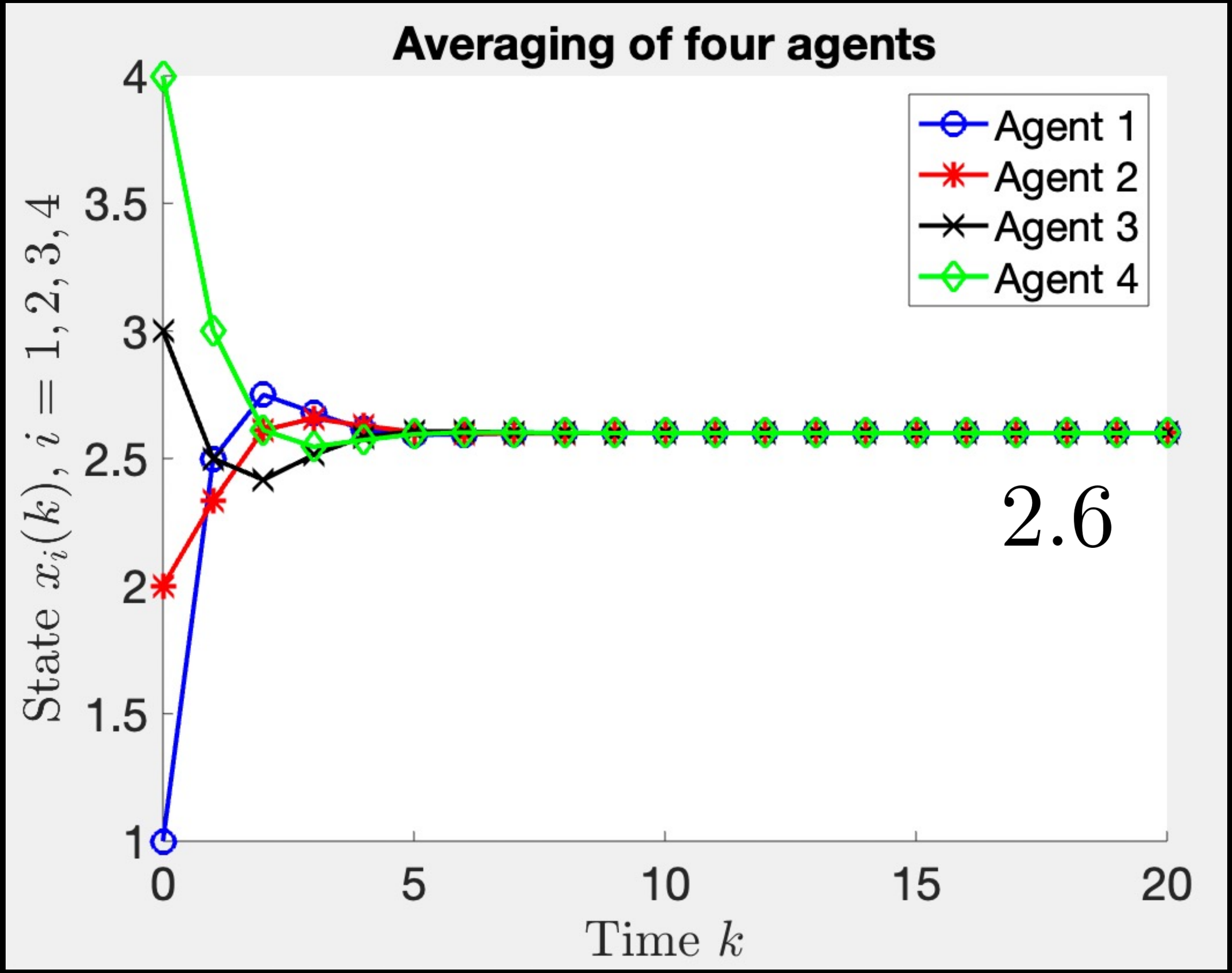


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$I - L$: not column stochastic

Example

$$\text{simulation: } x_1(0) = 1, x_2(0) = 2$$
$$x_3(0) = 3, x_4(0) = 4$$

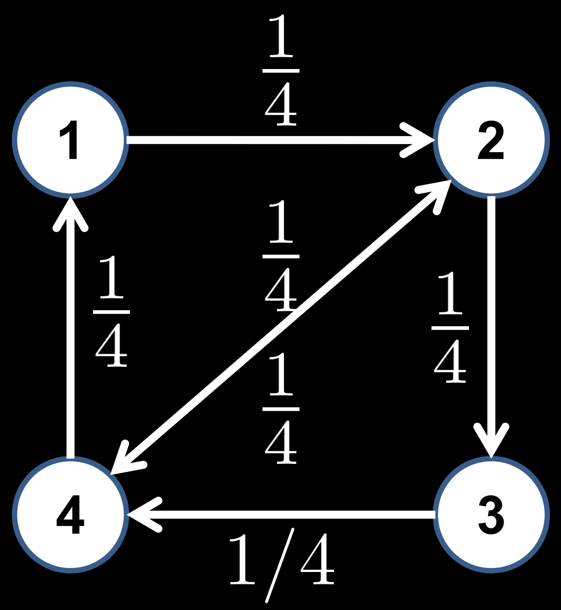


Example

example:

weighted graph $\mathcal{G}$

weight balanced

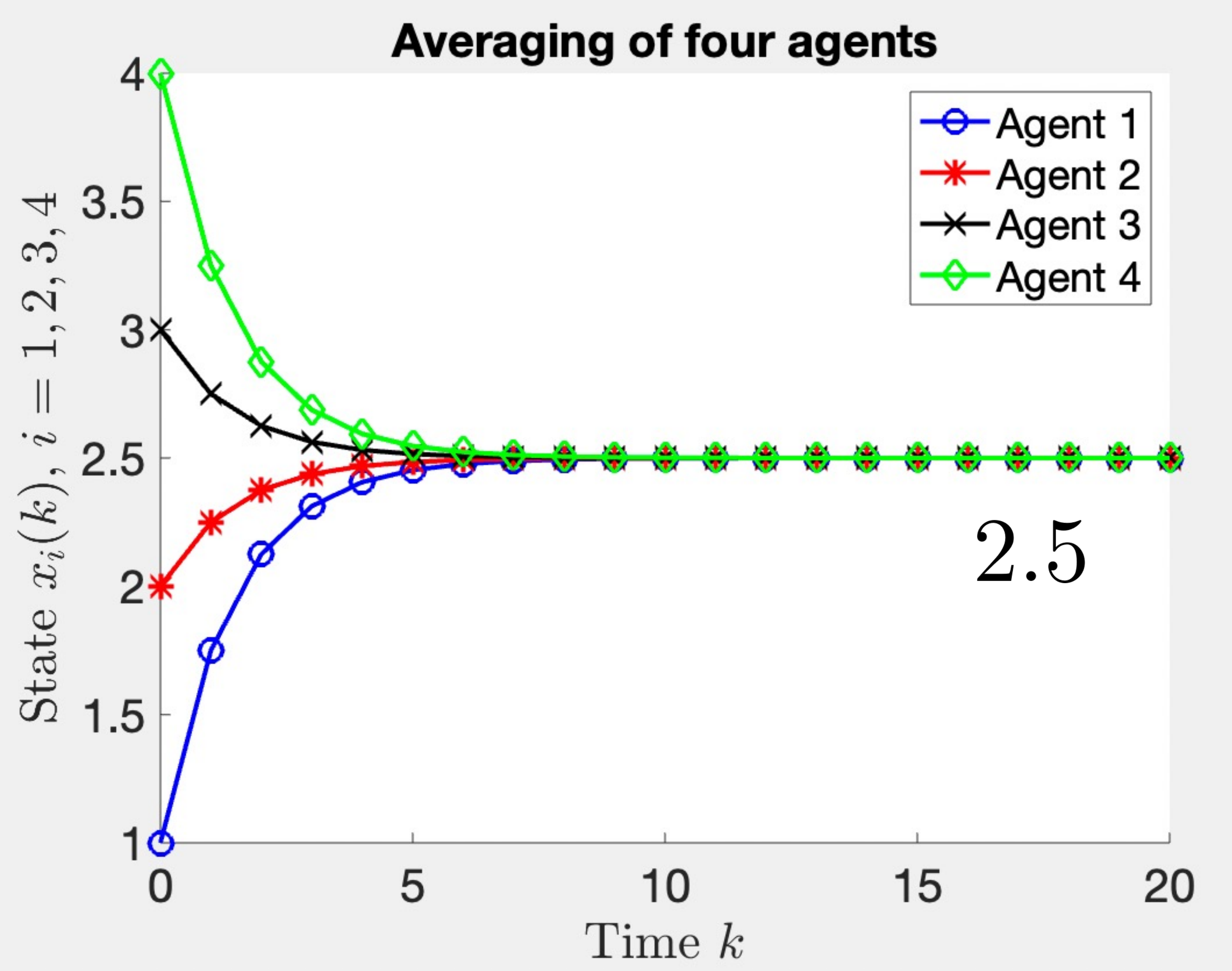


$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$I - L$: column stochastic

Example

$$\text{simulation: } x_1(0) = 1, x_2(0) = 2$$
$$x_3(0) = 3, x_4(0) = 4$$



Different weights

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

Distributed algorithm

$x_{i+1}(k) = x_i(k) + \frac{1}{n} \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k))$

$x(k+1) = (I - L)x(k)$ solves averaging

iff $\mathcal{G}$ is strongly connected and weight balanced

Different weights

a system of n interacting agents

is modeled by graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

each agent v_i has an initial value $x_i(0)$

Distributed algorithm

$$x_{i+1}(k) = x_i(k) + a_{ij} \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k))$$

$$(a_{ij} > 0, \sum_{j \in \mathcal{N}_i} a_{ij} < 1)$$

$x(k+1) = (I - L)x(k)$ solves averaging

iff $\mathcal{G}$ is strongly connected and weight balanced